LINEAR OPERATORS AND THE DISTRIBUTION

OF ZEROS OF ENTIRE FUNCTIONS

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By
Andrzej Piotrowski

Dissertation Committee:

George L. Csordas, Chairperson
David Bleecker
Thomas Craven
Thomas Hoover
Robert D. Little
Benjamin K. Bergen

We certify that we have read this dissertation and that, in our opinion, it is satisfactory in scope and quality as a dissertation for the degree of Doctor of Philosophy in Mathematics.

DISSERTATION COMMITTEE

Chairperson

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Abstract

If $\{\gamma_k\}_{k=0}^{\infty}$ is a sequence of real numbers and $Q = \{q_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials satisfying $\deg(q_k) = k$ for all non-negative integers k, then we can define a linear operator T_Q on the vector space of real polynomials by

$$T_Q[q_k(x)] = \gamma_k q_k(x) \qquad (k = 0, 1, 2, \dots).$$

If the linear operator T_Q has the property that it maps every real polynomial having only real zeros into another polynomial having only real zeros (or, perhaps, to the identically zero function), then the corresponding sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *Q-multiplier sequence*. Similarly, if the linear operator T_Q has the property that it does not increase the number of non-real zeros of any polynomial (which it does not map to the identically zero function), then the corresponding sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *Q-complex zero decreasing sequence*, or, for brevity, a *Q-CZDS*.

Pólya and Schur completely characterized all multiplier sequences for the standard basis $\{x^k\}_{k=0}^{\infty}$, which we will call the *classical multiplier sequences*. Turán, and subsequently Bleecker and Csordas, discovered classes of H-multiplier sequences, where H denotes the set of Hermite polynomials. In this dissertation, we completely characterize H-multiplier sequences and, therefore, solve an open problem stated in the literature six years ago. We show that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial H-multiplier sequence if and only if $\{\gamma_k\}_{k=0}^{\infty}$ is a classical multiplier sequence and, either $0 \leq \gamma_k \leq \gamma_{k+1}$, or $0 \geq \gamma_k \geq \gamma_{k+1}$ for all integers $k \geq 0$. In order to establish this result, we prove a significant generalization of a curve theorem due to Pólya.

In a series of papers, Craven and Csordas investigate CZDS for the standard basis $\{x^k\}_{k=0}^{\infty}$, which we will call the *classical CZDS*. We prove the existence of a large number of H-CZDS, where H denotes the set of Hermite polynomials. In order to do so, we generalize a result of Bleecker and Csordas, which itself is a generalization of a theorem due to Laguerre. We also demonstrate that the class of all polynomials which interpolate H-CZDS is the same as the class of all polynomials which interpolate classical CZDS.

Analogous results for other polynomial sets are also considered, including a class of generalized Hermite polynomials and the set of Laguerre polynomials. Furthermore, we prove that every Q-multiplier sequence (Q-CZDS) must be a classical multiplier sequence (classical CZDS), regardless of the choice of Q. Conversely, we show that, if every classical multiplier sequence is a Q-multiplier sequence, then there is a sequence of real numbers $\{c_k\}_{k=0}^{\infty}$ and a real constant β such that $Q = \{c_k\,(x+\beta)^k\}_{k=0}^{\infty}$.

The distribution of zeros of entire functions in strips in the complex plane is also considered. In this context, we generalize results due to Turán and obtain new sufficient conditions for the reality of zeros of polynomials in terms of the coefficients of their Hermite expansions.

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Chapter 1
Introduction

1.1 Historical Background and Motivation

Since ancient times, there has been great interest in solving algebraic equations. Indeed, clay tablets were discovered which demonstrate that the Babylonians knew of quadratic equations and some methods of their solution over 3500 years ago. Attempts to solve algebraic equations of higher degree have given rise to several important methods and mathematical constructs, the totality of which is often referred to as the *theory of equations*. Some aspects of the theory of equations have had far reaching consequences. For example, solving cubic and quartic equations required the manipulation of the square root of negative numbers, which led to the development of the complex number system. The theory of equations flourished in the hands of several prominent mathematicians including, but certainly not limited to, Descartes, Newton, Fourier, Gauss, Cauchy, Sturm, Hermite, Laguerre, Jensen, Pólya, Marden, and Turán.

Much of the development of the theory of equations regarding transcendental entire functions is a result of one of the most famous open problems in mathematics today. In 1859, Riemann studied the properties of a certain function which is now known as Riemann's zeta function $\zeta(z)$. This function is defined for $\operatorname{Re} z > 1$ by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and can be extended analytically to the entire complex plane, except for a simple pole at $z = 1$, and this extension is again denoted by $\zeta(z)$. It was hypothesized by Riemann that the non-trivial zeros of $\zeta(z)$ must lie on the critical line $\{z : \text{Im } z = 1/2\}$. Despite the work of many great mathematicians over the past century and a half, the validity of Riemann's hypothesis remains unknown. Riemann's hypothesis can be seen to be equivalent to the assertion that all of the zeros of the function $\xi(1/2 + iz)$ are real, where the entire function $\xi(z)$ is defined by

$$\xi(z) = \Gamma\left(\frac{z}{2} + 1\right)(z-1)\pi^{-z/2}\zeta(z)$$

and, as usual, $\Gamma(z)$ denotes the gamma function. Therefore, any results regarding necessary and/or sufficient conditions for an entire function to have only real zeros are of particular interest.

In studying the distribution of zeros of a function in a circular region in the complex plane, it is useful to examine the coefficients of the usual Vieta-Taylor expansion of the function. This is demonstrated by classical results due to many mathematicians, most notably Gauss, Cauchy, and Walsh (see, for example, [22]). In his 1950 paper *Sur l'algèbre fonctionnelle* [29], Turán was investigating the Riemann hypothesis and realized that, if one wanted to determine whether or not the zeros of a function lie in a certain strip in the complex plane which is symmetric about the real axis (in particular, whether or not all the zeros of a function are real), then one should expand the function in terms of Hermite polynomials. In light of this, Turán was able to take the aforementioned results of Gauss, Cauchy, and Walsh, and demonstrate analogous results concerning the relationship between the Hermite expansion coefficients of a function and its distribution of zeros in a strip (see [30]).

The study of the distribution of zeros of functions under the action of linear operators has also been an area of extensive research. For example, in 1691, Rolle demonstrated that, between any two real zeros of a differentiable real function $f(x)$, lies a zero of its derivative $f'(x)$. Thus, the movement of the zeros of a differentiable function $f(x)$ under the action of the linear operator $D = \dfrac{d}{dx}$ can, to a certain extent, be determined. For example, if we take $f(x)$ to be a polynomial having only real zeros, all of which lie in the interval $[a, b]$, then the zeros of $f'(x)$ are also real and lie in the interval $[a, b]$. Turning to the case where the zeros are not necessarily all real, it was shown by Lucas in 1874 that, if the zeros of a complex polynomial $p(x)$ lie in some convex polygon K in the complex plane, then the zeros of $p'(x)$ also lie in K. This result was also known to Gauss who mentioned it in the form of a mechanical interpretation of the zeros of the derivative (see [22, Preface]). Again, we see that the movement of the zeros of a complex polynomial under the action of the linear operator $D = \dfrac{d}{dz}$ is, in some sense, well-behaved.

If $\{\gamma_k\}_{k=0}^{\infty}$ is a sequence of real numbers, we can define a linear operator T on the vector space of real polynomials by

$$T[x^n] = \gamma_n x^n \qquad (n = 0, 1, 2, \dots). \tag{1.1}$$

Operators of this type have been studied by several authors. In particular, both

Laguerre [19] and Jensen [17] discovered a number of sequences $\{\gamma_k\}_{k=0}^{\infty}$ such that the corresponding operator T defined by (1.1) maps every polynomial which has only real zeros into another polynomial which has only real zeros. As a simple example, let us demonstrate that the sequence $\{k+1\}_{k=0}^{\infty}$ has this property. If the linear operator T is defined by (1.1), where $\gamma_k = k+1$, then it is easy to see that $T[p(x)] = \dfrac{d}{dx}\,(xp(x))$. Therefore, if $p(x)$ has only real zeros then, by Rolle's Theorem, $T[p(x)]$ also has only real zeros. In their 1914 paper [26], Pólya and Schur completely characterized all sequences with this property, which they called *multiplier sequences* (*of the first kind*).

In his 1950 paper [29], Turán announced an analogue of one of the results due to Laguerre alluded to in the previous paragraph. More precisely, if $\{\gamma_k\}_{k=0}^{\infty}$ is a sequence of real numbers, we can define a linear operator T_H on the vector space of real polynomials by its action on the Hermite polynomials

$$T_H[H_n(x)] = \gamma_n H_n(x) \qquad (n = 0, 1, 2, \dots).$$

Turán stated that the operator T_H corresponding to any sequence of the form $\{g(k)\}_{k=0}^{\infty}$, where $g(x)$ is a polynomial having only real negative zeros, takes every polynomial having only real zeros into another polynomial having only real zeros. In their 2001 paper [1], Bleecker and Csordas provided a proof and generalization of this result. These considerations led them to ask whether one could completely characterize all sequences with this property [1, Problem 4.1]. This problem has remained open until now and its complete solution, which appears in the last section of Chapter 5, is one of the main results contained in this dissertation.

In his 1916 paper [25], Pólya gave an amazing unification of three major theorems in the theory of the distribution of zeros of polynomials. This result made use of the Hermite-Poulain Theorem (a generalization of Rolle's Theorem) to demonstrate that a certain algebraic equation in two variables represents what Pólya termed an n^{th}-order curve. It is demonstrated that the curve must have n intersections with every line having a slope which is either non-negative or undefined. Since each intersection corresponds to a zero of a certain n^{th} degree polynomial, the conclusion of the theorem can be interpreted as a result regarding polynomials having only real zeros. This theorem has its shortcomings in that there are significant restrictions on the degree of the polynomial to be considered. However, we shall remedy this deficiency and also prove a more general curve theorem.

Multiplier sequences have been studied in great detail by several authors. In a series of papers [8]−[12], Craven and Csordas have given detailed accounts of the problems and theorems in the theory of multiplier sequences. In several of their papers, linear operators with another zero-mapping property are often considered. If the linear operator T, defined by

$$T[x^n] = \gamma_n x^n \qquad (n = 0, 1, 2, ...),$$

where $\{\gamma_k\}_{k=0}^{\infty}$ is a given sequence of real numbers, has the property that it does not increase the number of *non-real* zeros of any real polynomial, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *complex zero decreasing sequence* or, for brevity, a CZDS. In particular, every CZDS must be a multiplier sequence. However, it is somewhat surprising that there are multiplier sequences which are *not* CZDS (see [11, Example 1.8]). We note that, in contrast to multiplier sequences, there is no known characterization of CZDS. To further underscore the importance of linear operators in the theory of distribution of zeros of entire functions, we mention that several linear operators, such as multiplier sequences, CZDS, differentiation $\left(D = \dfrac{d}{dx}\right)$, and $\exp(\lambda D^2)$, have been used by several authors, including Pólya, DeBruijn, Csordas, Smith, and Varga, to study the Riemann Hypothesis.

1.2 A Brief Synopsis

Chapter 2 consists primarily of background information and notation involving the Hermite polynomials, a class of generalized Hermite polynomials, the Laguerre-Pólya class, and some well-known theorems (and their consequences) regarding the distribution of zeros of entire functions.

Next, the study of linear operators on entire functions begins with the establishment of several operator identities, both known and new (see, in particular, Proposition 33), which will be used in the sequel. Classical results regarding linear operators with certain zero-mapping properties are surveyed, and a theorem due to Laguerre, which was generalized by Bleecker and Csordas, is further generalized to demonstrate the existence of a previously unknown complex zero decreasing operator (Proposition 52). Chapter 3 concludes with the extension of results concerning linear operators on polynomials to linear operators on transcendental entire functions.

Chapter 4 is devoted to the study of the distribution of zeros of Hermite expansions. The operator $\exp(-tD^2)$ is employed to prove results regarding zeros in a strip which are analogous to classical results regarding zeros in a circle (Corollaries 79, 81, and 83). A result of Turán is improved upon (Proposition 93 and Remark 94), and it is shown that there is a limit to which this result can be extended (Proposition 88).

Turning to linear operators defined by their action on the Hermite polynomials, several classes of H-CZDS are displayed, all of which are new (Theorems 101 and 104), polynomials which interpolate H-CZDS are characterized (111), and new classes of H-multiplier sequences are also given (Proposition 116 and Remark 117). Connections between classical and Hermite multiplier sequences and CZDS are exhibited (Propositions 109 and 118), and, in particular, it is shown that every non-trivial non-negative H-multiplier sequence must be a non-decreasing multiplier sequence (Theorem 127).

The majority of Chapter 5 is dedicated to Pólya's curve theorem (Theorem 136) and our generalization of this theorem (Theorem 147), which is used to completely characterize all H-multiplier sequences (Theorem 152 and Remark 153).

In Chapter 6, Q-multiplier sequences and Q-CZDS are investigated, where Q is an arbitrary simple set of polynomials. In particular, we obtain several results when we take Q to be the generalized Hermite polynomials of Chapter 2 (Section 6.2) and also when we take Q to be the set of Laguerre polynomials (Section 6.3). In the general setting, it is shown that every Q-multiplier sequence (Q-CZDS) must be a classical multiplier sequence (classical CZDS), regardless of the choice of Q (Theorems 158 and 159). Conversely, it is shown that if every classical multiplier sequence is a Q-multiplier sequence, then $Q = \{c_k\,(x+\beta)^k\}_{k=0}^{\infty}$, where $\{c_k\}_{k=0}^{\infty}$ is a sequence of real numbers and $\beta \in \mathbb{R}$.

Chapter 2
Polynomials and Transcendental Entire Functions

2.1 Hermite Polynomials

We will make frequent use the of the Hermite polynomials $\{H_k(x)\}_{k=0}^{\infty}$ which are defined by the generating relation

$$\exp(2xt - t^2) = \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} t^k, \tag{2.1}$$

which is valid for all $x, t \in \mathbb{C}$. Let us first follow Rainville [27, p. 189] in obtaining an explicit (Rodrigues) formula for $H_n(x)$. By Maclaurin's theorem, we have

$$H_n(x) = \left[\frac{d^n}{dt^n} e^{2xt - t^2} \right]_{t=0}.$$

Multiplying by e^{-x^2} and substituting $w = x - t$, we have

$$e^{-x^2} H_n(x) = \left[\frac{d^n}{dt^n} e^{-(x-t)^2} \right]_{t=0} = (-1)^n \left[\frac{d^n}{dw^n} e^{-w^2} \right]_{w=x} = (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Thus, the Hermite polynomials can be explicitly defined by the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2) \qquad (n = 0, 1, 2, \dots). \tag{2.2}$$

Alternatively, one could examine the generating relation (2.1) to obtain the formula (see [27, p. 187])

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!}. \tag{2.3}$$

For the convenience of the reader, the first few Hermite polynomials are listed here.

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \\ H_4(x) &= 16x^4 - 48x^2 + 12, \\ H_5(x) &= 32x^5 - 160x^3 + 120x. \end{aligned}$$

By equation (2.3) we see that, for each $n = 0, 1, 2, \dots$, the degree of $H_n(x)$ is precisely n. Thus, the Hermite polynomials form a basis for the vector space of real polynomials $\mathbb{R}[x]$. Furthermore, equation (2.3) also shows that

$$H_n(-x) = (-1)^n H_n(x) \qquad (n = 0, 1, 2, \dots). \tag{2.4}$$

Differentiating the generating relation (2.1) with respect to x, we obtain the relation

$$H_n'(x) = 2nH_{n-1}(x) \qquad (n = 1, 2, 3, \dots). \tag{2.5}$$

Similarly, differentiating the generating relation (2.1) with respect to t, we obtain the pure recurrence relation

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \qquad (n = 2, 3, 4, \dots). \tag{2.6}$$

Combining the relations (2.5) and (2.6) we obtain Hermite's differential equation

$$nH_n(x) = xH_n'(x) - \frac{1}{2}H_n''(x) \qquad (n = 0, 1, 2, \dots), \tag{2.7}$$

which will play an important role in the following chapters.

One can use Hermite's differential equation and the Rodrigues formula to show (see [27, pp. 192])

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2^n n! \sqrt{\pi} & \text{if } m = n. \end{cases} \tag{2.8}$$

Thus, the Hermite polynomials form an orthogonal set over the interval $(-\infty, \infty)$ with respect to the weight function $\exp(-x^2)$. Therefore, the well-known results about orthogonal polynomials ([27, Chapter 9]) apply to the Hermite polynomials. In particular, for each n, $H_n(x)$ has only simple real zeros, and the Hermite polynomials satisfy the Christoffel-Darboux formula (see [27, p. 154 and p. 193])

$$\sum_{k=0}^{n} \frac{H_k(x) H_k(y)}{2^k k!} = \frac{H_{n+1}(y) H_n(x) - H_{n+1}(x) H_n(y)}{2^{n+1} n! (y-x)}. \tag{2.9}$$

There is an interesting formula for the product of two Hermite polynomials which will also be of interest (see, for example, [6]).

$$H_m(x) H_n(x) = \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(x). \tag{2.10}$$

We will also make use of a class of generalized Hermite polynomials

$$\mathcal{H}^{\alpha} = \left\{ \mathcal{H}_n^{(\alpha)}(x) \right\}_{k=0}^{\infty},$$

which depend on a real parameter α. We define these polynomials by the generating relation

$$\exp\left(xt - \frac{\alpha}{2} t^2 \right) = \sum_{k=0}^{\infty} \frac{\mathcal{H}_k^{(\alpha)}(x)}{k!} t^k \qquad (\alpha \in \mathbb{R}), \tag{2.11}$$

which is valid for all $x, t \in \mathbb{C}$.

Remark 1. If $\alpha = 0$, then these generalized Hermite polynomials simply reduce to the standard basis $\mathcal{H}^0 = \left\{ \mathcal{H}_n^{(0)}(x) \right\}_{k=0}^{\infty} = \left\{ x^k \right\}_{k=0}^{\infty}$. It should also be noted that some authors actually define the Hermite polynomials to be the sequence $\mathcal{H}^2 = \left\{ H_k^{(2)}(x) \right\}_{k=0}^{\infty}$ (see, for example, [3]).

In the case where $\alpha \neq 0$, we again use Maclaurin's theorem and the substitution $w = x - \alpha t$ to see that

$$\begin{aligned} \exp\left(-\frac{x^2}{2\alpha}\right) \mathcal{H}_n^{(\alpha)}(x) &= \left[\frac{d^n}{dt^n} \exp\left(-\frac{1}{2\alpha}(x-\alpha t)^2\right)\right]_{t=0} \\ &= (-\alpha)^n \left[\frac{d^n}{dw^n} \exp\left(-\frac{w^2}{2\alpha}\right)\right]_{w=x} \\ &= (-\alpha)^n \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2\alpha}\right). \end{aligned}$$

Thus, for $\alpha \neq 0$, the generalized Hermite polynomials $\mathcal{H}^\alpha$ can be explicitly defined by the Rodrigues formula

$$\mathcal{H}_n^{(\alpha)}(x) = (-\alpha)^n \exp\left(\frac{x^2}{2\alpha}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2\alpha}\right) \qquad \left(\alpha \in (\mathbb{R} \setminus \{0\})\, ;\, n = 0, 1, 2, \ldots\right). \tag{2.12}$$

We may now obtain, for $\alpha \neq 0$, a relation between the generalized Hermite polynomials $\mathcal{H}^\alpha$ and the classical Hermite polynomials H. For any differentiable function $f(x)$ and any non-zero real number a, we have

$$\left(\frac{1}{a}\right)^n \frac{d^n}{dx^n} f(ax) = f^{(n)}(ax) = \left[\frac{d^n}{dw^n} f(w)\right]_{w=ax}.$$

Thus

$$\begin{aligned} \mathcal{H}_n^{(\alpha)}(\sqrt{2\alpha}\, x) &= (-\alpha)^n \exp\left(x^2\right) \left[\frac{d^n}{dw^n} \exp\left(-\frac{w^2}{2\alpha}\right)\right]_{w=\sqrt{2\alpha}\, x} \\ &= (-\alpha)^n \exp(x^2) \left(\frac{1}{\sqrt{2\alpha}}\right)^n \frac{d^n}{dx^n} \exp(-x^2) \\ &= \left(\frac{\alpha}{2}\right)^{n/2} H_n(x), \end{aligned}$$

which implies the relation

$$\mathcal{H}_n^{(\alpha)}(x) = \left(\frac{\alpha}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2\alpha}}\right) \qquad \big(\alpha \in (\mathbb{R} \setminus \{0\})\,;\; n = 0, 1, 2, \ldots\big). \tag{2.13}$$

Relation (2.13) may be used to obtain the explicit formula

$$\mathcal{H}_n^{(\alpha)}(x) = \sum_{k=0}^{[n/2]} \frac{(-\alpha)^k n! x^{n-2k}}{2^k k!(n-2k)!}, \tag{2.14}$$

the recurrence relation

$$\mathcal{H}_n^{(\alpha)}(x) = x\mathcal{H}_{n-1}^{(\alpha)}(x) - \alpha(n-1)\mathcal{H}_{n-2}^{(\alpha)}(x) \qquad (\alpha \in \mathbb{R};\; n = 2, 3, 4, \ldots), \tag{2.15}$$

and the differential equations

$$\frac{d}{dx}\mathcal{H}_n^{(\alpha)}(x) = n\mathcal{H}_{n-1}^{(\alpha)}(x) \qquad (\alpha \in \mathbb{R};\; n = 1, 2, 3, \ldots), \tag{2.16}$$

$$n\mathcal{H}_n^{(\alpha)}(x) = x\frac{d}{dx}\mathcal{H}_n^{(\alpha)}(x) - \alpha\frac{d^2}{dx^2}\mathcal{H}_n^{(\alpha)}(x) \qquad (\alpha \in \mathbb{R};\; n = 0, 1, 2, \ldots). \tag{2.17}$$

For the convenience of the reader, we list the first few generalized Hermite polynomials here.

$$\begin{aligned}
\mathcal{H}_0^{(\alpha)}(x) &= 1, \\
\mathcal{H}_1^{(\alpha)}(x) &= x, \\
\mathcal{H}_2^{(\alpha)}(x) &= x^2 - \alpha, \\
\mathcal{H}_3^{(\alpha)}(x) &= x^3 - 3\alpha x, \\
\mathcal{H}_4^{(\alpha)}(x) &= x^4 - 6\alpha x^2 + 3\alpha^2, \\
\mathcal{H}_5^{(\alpha)}(x) &= x^5 - 10\alpha x^3 + 15\alpha^2 x.
\end{aligned}$$

For $\alpha > 0$, we may employ relation (2.13) to see that the generalized Hermite polynomials $\mathcal{H}^\alpha$ satisfy

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\alpha}\right) \mathcal{H}_n^{(\alpha)}(x)\mathcal{H}_m^{(\alpha)}(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \text{ and } \alpha > 0, \\ \alpha^n n!\sqrt{2\pi\alpha} & \text{if } m = n \text{ and } \alpha > 0. \end{cases}$$

Thus, for $\alpha > 0$, the generalized Hermite polynomials $\mathcal{H}^\alpha$ form an orthogonal set over the interval $(-\infty, \infty)$ with respect to the weight function $\exp\left(-\dfrac{x^2}{2\alpha}\right)$. However, if $\alpha \leq 0$, then the generalized Hermite polynomials $\mathcal{H}^\alpha$ do *not* form an orthogonal set over any real interval. Indeed, every polynomial in an orthogonal set must have only simple real zeros (see [27, p. 149]), but the polynomial $\mathcal{H}_2^{(\alpha)}(x) = x^2 - \alpha$ has a zero of multiplicity 2 when $\alpha = 0$ and has two non-real zeros whenever $\alpha < 0$.

We desire to prove an addition formula for the generalized Hermite polynomials $\mathcal{H}^\alpha$. As we will see, the addition formula applies to a wider class of polynomials to which these generalized Hermite polynomials belong.

Definition 2. A sequence of polynomials $\{p_k(x)\}_{k=0}^\infty$ is called an *Appell sequence* if $p_0(x)$ is a non-zero constant and

$$p_n'(x) = np_{n-1}(x) \qquad (n = 1, 2, 3, \dots). \tag{2.18}$$

There are several easily deduced necessary and sufficient conditions for a sequence of polynomials to be an Appell sequence. One such condition involves an addition formula which will be pertinent to our later investigations. The following proposition is known, but in the absence of a good reference, we include its proof for the sake of completeness.

Proposition 3. *Let $P = \{p_k(x)\}_{k=0}^\infty$ be a sequence of polynomials and suppose $p_0(x)$ is a non-zero constant function. Then P is an Appell sequence if and only if the addition formula*

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(x) y^k \tag{2.19}$$

holds for every non-negative natural number n.

Proof. Suppose that the addition formula (2.19) holds for every non-negative integer n. Then, in particular,

$$p_n(y) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(0) y^k \qquad (n = 0, 1, 2, \ldots).$$

Thus, for any non-negative integer n,

$$p_n'(y) = \sum_{k=1}^{n} \binom{n}{k} p_{n-k}(0) k y^{k-1} = n \sum_{k=0}^{n-1} \binom{n-1}{k} p_{n-1-k}(0) y^k = n p_{n-1}(y).$$

Therefore P is an Appell sequence.

Conversely, suppose P is an Appell sequence. We shall prove by induction that the addition formula (2.19) holds for every non-negative integer n. Since $p_0(x)$ is assumed to be a non-zero constant function, the addition formula (2.19) clearly holds for $n = 0$. Fix $n \geq 1$ and suppose the addition formula (2.19) holds for $p_{n-1}(x)$. Then, for any fixed $x \in \mathbb{R}$,

$$\begin{aligned} \int n p_{n-1}(x+y)\, dy &= \int n \sum_{k=0}^{n-1} \binom{n-1}{k} p_{n-1-k}(x) y^k \, dy \\ &= \sum_{k=0}^{n-1} n \binom{n-1}{k} p_{n-1-k}(x) \frac{y^{k+1}}{k+1} + c \\ &= \sum_{k=1}^{n} \binom{n}{k} p_{n-k}(x) y^k + c. \end{aligned}$$

Since P is an Appell sequence and differentiation is translation invariant, we have, for each fixed $x \in \mathbb{R}$,

$$\frac{d}{dy}p_n(x+y) = np_{n-1}(x+y)$$

Thus, there exists a constant c such that

$$p_n(x+y) = \sum_{k=1}^{n} \binom{n}{k} p_{n-k}(x)y^k + c.$$

The addition formula (2.19) now follows from the relation $c = p_n(x+0) = p_n(x)$. $\square$

The relation (2.16) shows that, for each $\alpha \in \mathbb{R}$, the generalized Hermite polynomials $\mathcal{H}^\alpha$ form an Appell sequence. Whence, by Proposition 3, the generalized Hermite polynomials $\mathcal{H}^\alpha$ satisfy the addition formula

$$\mathcal{H}_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{(\alpha)}(x)y^k \qquad (\alpha \in \mathbb{R};\ n = 0, 1, 2, \dots). \tag{2.20}$$

Incidentally, the addition formula for the generalized Hermite polynomials, together with the relation between the generalized and classical Hermite polynomials (2.13), gives rise to an addition formula for the classical Hermite polynomials

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x)(2y)^k \qquad (n = 0, 1, 2, \dots),$$

which is stated in, e.g., [15, p. 432].

2.2 Zeros of Polynomials

As is usually customary, we will call a complex number z_0 a *zero* of the complex function $f(z)$ if $f(z_0) = 0$. In this situation, we will also say that z_0 is a *root* of the equation $f(z) = 0$. One of the most important results regarding the zeros of polynomials is the Fundamental Theorem of Algebra.

Theorem 4. (The Fundamental Theorem of Algebra) *Let $p(z)$ be a complex polynomial of degree $n \geq 1$. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

By repeated application of the Fundamental Theorem of Algebra, we have the following important result.

Corollary 5. *Every complex polynomial of degree $n \geq 1$ has exactly n complex zeros, counting multiplicities.*

There are many interesting connections between the zeros of a function and its derivative. One particularly interesting and useful result along these lines is the theorem of Rolle, which one generally learns in an introductory calculus class.

Theorem 6. (Rolle's Theorem) *Suppose $f(x)$ is a continuous function on the interval $[a, b]$ which is differentiable on the interval (a, b). If $f(a) = f(b)$, then there exists a number c in the interval (a, b) such that $f'(c) = 0$. In particular, if a and b are zeros of $f(x)$, then there is a zero of $f'(x)$ which lies between a and b.*

The following corollary will be used frequently in the following chapters.

Corollary 7. *Suppose $f(x)$ is a continuous function on the interval $[a, b]$ which is differentiable on the interval (a, b). If $f(x)$ has exactly m zeros, counting multiplicities, in the interval $[a, b]$, then $f'(x)$ has at least $m - 1$ zeros, counting multiplicities, in the interval $[a, b]$.*

Proof. Let $x_1 < x_2 < x_3 < \cdots < x_j$ be the distinct zeros of $f(x)$ in the interval $[a, b]$ of multiplicities $m_1, m_2, m_3, \ldots, m_j$, respectively. Then we have

$$m = m_1 + m_2 + m_3 + \cdots + m_j$$

Each zero x_i of $f(x)$ is a zero of $f'(x)$ of multiplicity $m_i - 1$, which accounts for $m - j$ zeros of $f'(x)$ in the interval $[a, b]$. By Rolle's theorem, for each $i = 1, 2, 3, \ldots, j - 1$, there is at least one zero of $f'(x)$ in each of the intervals (x_i, x_{i+1}). Therefore $f'(x)$, which is of degree $m - 1$, has at least $m - j + (j - 1) = m - 1$ zeros in the interval $[a, b]$. □

A useful tool in examining the distribution of zeros of entire functions is due to Rouché. This theorem takes on many forms in the literature, but, for our purposes, we only require the following version.

Theorem 8. (Rouché's Theorem. [22, p.2]) *If $P(z)$ and $Q(z)$ are analytic interior to a simple closed curve C and if they are continuous on C and*

$$|P(z) - Q(z)| < |Q(z)| \tag{2.21}$$

for all $z \in C$, then $P(z)$ has the same number of zeros interior to C as does $Q(z)$, counting multiplicities.

One can see the beauty of this theorem in the way it easily yields a proof of Corollary 5. Indeed, given any complex polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$, we let $Q(z) = a_n z^n$. For all sufficiently large values of R, one can show that (2.21) holds for all z on the circle centered at the origin of radius R. Thus, inside each one of these circles, $P(z)$ has the same number of zeros as $Q(z)$, which has only one zero at the origin of multiplicity n.

We will be investigating properties of polynomials and, whenever possible, we will want to extend these properties to a more general class of entire functions. Appropriate here is the notion of uniform convergence on compact subsets of $\mathbb{C}$.

Definition 9. A sequence of entire functions $\{f_n(z)\}_{n=0}^{\infty}$ is said to *converge uniformly on compact subsets of* $\mathbb{C}$ *to the function* $f(z)$ if, for every compact subset $K \subset \mathbb{C}$ and every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies that $|f(z) - f_n(z)| \leq \epsilon$ for all $z \in K$.

One of the pleasant aspects of uniform convergence on compact subsets is that the limit function is guaranteed to be an entire function.

Theorem 10. ([28, Theorem 10.28]) *If the sequence of entire functions* $\{f_n(z)\}_{n=0}^{\infty}$ *converge uniformly on compact subsets of* $\mathbb{C}$ *to the function* $f(z)$*, then* $f(z)$ *is an entire function and the sequence of functions* $\{f_n'(z)\}_{n=0}^{\infty}$ *converge uniformly on compact subsets of* $\mathbb{C}$ *to* $f'(z)$*.*

We will want to extend certain results regarding the zeros of polynomials to transcendental entire functions. In this context, the following theorem of Hurwitz is essential.

Theorem 11. (Hurwitz' Theorem [22, p. 4]) *Suppose the sequence of entire functions* $\{f_n(z)\}_{n=0}^{\infty}$ *converge uniformly on compact subsets of* $\mathbb{C}$ *to the function* $f(z)$*, where* $f(z)$ *is not identically zero. If* $z_0 \in \mathbb{C}$ *is a limit point of the zeros of the functions* $f_n(z)$*, then* z_0 *is a zero of* $f(z)$*. Conversely, if* $z_0 \in \mathbb{C}$ *is a zero of* $f(z)$ *of multiplicity* m*, then, for every sufficiently small neighborhood* K *of* z_0*, there exists an integer*

$N = N(K)$ such that K contains exactly m zeros of $f_n(z)$ (counting multiplicities) whenever $n \geq N$.

A real number can be the limit of a sequence of non-real numbers, but a non-real number *cannot* be the limit of a sequence of real numbers. Thus, as a consequence of Hurwitz' theorem, we have the following corollary.

Corollary 12. *Suppose the sequence of entire functions $\{f_n(z)\}_{n=0}^{\infty}$ converges uniformly on compact subsets of $\mathbb{C}$ to the function $f(z)$, which is not identically zero. Then there exists an integer N such that the number of non-real zeros of $f(z)$ (counting multiplicities) is less than or equal to the number of non-real zeros of $f_n(z)$ (counting multiplicities) whenever $n \geq N$. In particular, if $f(z)$ is the uniform limit on compact subsets of $\mathbb{C}$ of entire functions having only real zeros and if $f(z)$ is not identically zero, then $f(z)$ has only real zeros.*

Finally, we give here a sufficient condition for uniform convergence on compact subsets which will be tacitly used throughout the following chapters.

Proposition 13. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial. If the coefficients of the complex polynomials*

$$p_j(z) = \sum_{k=0}^{n} a_{k,j} z^k \quad (j = 1, 2, 3, \ldots)$$

satisfy the condition

$$\lim_{j\to\infty} a_{k,j} = a_k \qquad (k = 0, 1, 2, \ldots, n)$$

then the sequence of polynomials $\{p_j(z)\}_{j=0}^{\infty}$ converge uniformly on compact subsets of $\mathbb{C}$ to $p(z)$.

Proof. Fix a compact subset K of $\mathbb{C}$ and let $\epsilon > 0$ be given. Set $M = \max\left\{1, \sup_{z\in K}|z|\right\}$.

For each k choose N_k so that

$$|a_k - a_{k,j}| < \frac{\epsilon}{(n+1)M^n}$$

whenever $j \geq N_k$, and set $N = \max\{N_0, N_1, \ldots, N_n\}$. Then, for $j \geq N$ and $z \in K$,

$$\begin{aligned}
|p(z) - p_j(z)| &= \left|\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n} a_{k,j} z^k\right| \\
&= \left|\sum_{k=0}^{n} (a_k - a_{k,j}) z^k\right| \\
&\leq \sum_{k=0}^{n} |a_k - a_{k,j}||z|^k \\
&< \sum_{k=0}^{n} \frac{\epsilon}{(n+1)M^n} M^n = \epsilon.
\end{aligned}$$

□

Remark 14. It should be noted that, in Proposition 13, we did not assume that any of the coefficients a_k were non-zero. Thus, for example, Proposition 13 implies that the sequence of polynomials $\left\{\frac{x}{k}\right\}_{k=0}^{\infty}$ converge uniformly to the identically zero function. This example demonstrates that, in Hurwitz's theorem (Theorem 11), and also in Corollary 12, the requirement that the limit function not be identically zero is necessary.

2.3 The Laguerre-Pólya Class

As was already mentioned, we will be investigating the distribution of zeros of polynomials and, whenever possible, we would also like to extend our considerations to transcendental entire functions. In light of Hurwitz' theorem, the notion of uniform convergence will play a significant role. To each entire function, there is a certain sequence of polynomials, called Jensen polynomials, which arise naturally in this setting.

Definition 15. Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k$ be an arbitrary entire function. Then the n^{th} *Jensen polynomial associated with the function* $\varphi(x)$ is defined by

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \alpha_k x^k \qquad (n = 0, 1, 2, \dots).$$

The Jensen polynomials associated with a given entire function satisfy a large number of important properties (see [9]). In particular, Jensen polynomials can be used to approximate entire functions, which is demonstrated by the following lemma.

Lemma 16. (Craven-Csordas [9, Lemma 2.2]) *Let* $\varphi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k$ *be an arbitrary entire function and let* $\{g_n(x)\}_{n=0}^{\infty}$ *be the Jensen polynomials associated with* $\varphi(x)$*. Then the sequence of polynomials* $\left\{ g_n\left(\frac{x}{n}\right)\right\}_{n=0}^{\infty}$ *converges uniformly on compact subsets of* $\mathbb{C}$ *to* $\varphi(x)$*.*

Real entire functions which are the uniform limit on compact subsets of $\mathbb{C}$ of polynomials having all there zeros in some prescribed region have been studied by several authors (see, for example, [12] and the references contained therein). In particular, if each of the approximating polynomials has only real zeros, then the given entire function must be of a very specific form.

Definition 17. A real entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to belong to the *Laguerre-Pólya class*, written $\varphi \in \mathcal{L} - \mathcal{P}$, if it can be written in the form

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k}$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \geq 0$, $0 \leq \omega \leq \infty$, and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$.

Remark 18. A real entire function $\varphi(x)$ belongs to the Laguerre-Pólya class if and only if it is the uniform limit on compact subsets of $\mathbb{C}$ of real polynomials having only real zeros (See, for example, [20, Ch. VIII] or [23, Satz 9.2]).

Notation 19. If $-\infty \leq a < b \leq \infty$ and the zeros of $\varphi(x) \in \mathcal{L} - \mathcal{P}$ all lie in the interval (a, b), $(a, b]$, $[a, b)$, or $[a, b]$, then we will write $\varphi(x) \in \mathcal{L} - \mathcal{P}(a, b)$, $\varphi(x) \in \mathcal{L} - \mathcal{P}(a, b]$, $\varphi(x) \in \mathcal{L} - \mathcal{P}[a, b)$, or $\varphi(x) \in \mathcal{L} - \mathcal{P}[a, b]$, respectively.

Definition 20. A real entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to be of *type I in the Laguerre-Pólya class*, written $\varphi(x) \in \mathcal{L} - \mathcal{P}I$, if $\varphi(x)$ or $\varphi(-x)$ can be written in the form

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right),$$

where $c \in \mathbb{R}$, m is a non-negative integer, $\sigma \geq 0$, $x_k > 0$, $0 \leq \omega \leq \infty$, and $\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$.

Remark 21. A real entire function $\varphi(x)$ is of type I in the Laguerre-Pólya class if and only if it is the uniform limit on compact subsets of $\mathbb{C}$ of real polynomials having only real zeros, all of which have the same sign (see, for example, [20, Chapter VIII] or [23, Satz 9.1]). Furthermore, an entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}$ is of type I in the Laguerre-Pólya class if and only if either $\gamma_k \geq 0$, $-\gamma_k \geq 0$, $(-1)^k\gamma_k \geq 0$, or $(-1)^{k+1}\gamma_k \geq 0$ for all non-negative integers k.

Notation 22. If $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}$ and the Taylor coefficients γ_k of $\varphi(x)$ are all non-negative, then we will write $\varphi \in \mathcal{L} - \mathcal{P}^+$.

As the next lemma demonstrates, one can determine whether or not the Taylor coefficients of a function in the class $\mathcal{L} - \mathcal{P}I$ are non-decreasing by examining the product representation of the function. This fact will turn out to be very important in some of our later investigations.

Lemma 23. (Craven-Csordas [8, Lemma 2.2]) *Let* $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *be a transcendental entire function of type I in the Laguerre-Pólya class with the product representation*

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right),$$

where $c \in \mathbb{R}$, m *is a non-negative integer,* $\sigma \geq 0$, $x_k > 0$, $0 \leq \omega \leq \infty$, *and* $\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$. *Then* $\sigma \geq 1$ *if and only if* $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$.

Chapter 3
Linear Operators on Real Polynomials

3.1 Notation and Operator Identities

Let $D = \dfrac{d}{dx}$ denote differentiation with respect to x. In general, if

$$\psi(y) = \sum_{k=0}^{\infty} p_k(x) y^k \qquad \big(p_k(x) \in \mathbb{C}[x];\ k = 0, 1, 2, \ldots\big)$$

is a formal power series, then we define the linear operator $\psi(D)$ by

$$\psi(D)[f(x)] = \sum_{k=0}^{\infty} p_k(x) f^{(k)}(x) \tag{3.1}$$

whenever the right hand side of (3.1) represents an analytic function in some neighborhood of the origin. In the case where each of the polynomials $p_k(x)$ are real constants, the operator $\psi(D)$ has been studied by several authors (see [10] and the references therein). When $f(x)$ is a polynomial, the right hand side of (3.1) is again a polynomial and so the question of convergence does not arise.

We will often think of operators of the form (3.1) as objects in themselves

$$\psi(D) = \sum_{k=0}^{\infty} p_k(x) D^k,$$

where we take D^0 to be the identity operator I. Furthermore, we shall often suppress the symbol I. For example, the operator $(I + xD)$ will simply be written as $(1 + xD)$. Also, when applying several operators in a row, we will adopt the convention of applying the operators in order from right to left. For example,

$$D(x-D)[f(x)] = D[xf(x) - f'(x)] = xf'(x) + f(x) - f''(x).$$

This convention is important since, in general, two operators need not commute. For example,

$$\big(xD\big)\big[f(x)\big] = xf'(x)$$

while

$$\big(Dx\big)\big[f(x)\big] = D\big[xf(x)\big] = f(x) + xf'(x). \tag{3.2}$$

Thus, the operators $x = xI$ and D do not commute. However, equation (3.2) suggests that the operators Dx and $(1 + xD)$ are actually the same operator. In general, two operators T_1 and T_2 are equal if they have the same domain and range and, for every element v of the domain, $T_1[v] = T_2[v]$. We will now demonstrate equality between certain operators which will be of importance in the sequel.

Lemma 24. *For any non-negative integer m and any entire function $f(x)$,*

$$D^m xD[f(x)] = \big(xD^{m+1} + mD^m\big)[f(x)], \tag{3.3}$$

where D denotes differentiation with respect to x.

Proof. From Leibniz' formula for the n^{th} derivative of the product of two functions

$$D^n[f(x)g(x)] = \sum_{k=0}^{n} \binom{n}{k} D^k[f(x)]D^{n-k}[g(x)],$$

we have

$$D^m xD[f(x)] = D^m[xf'(x)] = \sum_{k=0}^{n} \binom{m}{k} D^k[x]D^{m-k}[f'(x)] = xf^{(m+1)}(x) + mf^{(m)}(x).$$

□

Lemma 25. *Let m be a non-negative integer, $\beta \in \mathbb{R}$, and let $f(x)$ be an entire function. Then*

$$D^m\big(\delta_\beta - r\big)[f(x)] = \big(\delta_\beta - (r-m)\big)D^m[f(x)], \tag{3.4}$$

where $\delta_\beta = xD - \beta D^2$.

Proof. Examination of the left hand side of equation (3.4) yields

$$\begin{aligned} D^m\big(\delta_\beta - r\big)[f(x)] &= D^m\big[xD[f(x)] - \beta D^2[f(x)] - r[f(x)]\big] \\ &= D^m xD[f(x)] - \beta D^{m+2}[f(x)] - rD^m[f(x)]. \end{aligned}$$

Therefore, by Lemma 24,

$$\begin{aligned} D^m\big(\delta_\beta - r\big)[f(x)] &= \big(xD^{m+1}[f(x)] + mD^m[f(x)]\big) - \beta D^{m+2}[f(x)] - rD^m[f(x)] \\ &= \big(xD - \beta D^2 + m - r\big)\, D^m[f(x)] \\ &= \big(\delta_\beta - (r-m)\big)D^m[f(x)]. \end{aligned}$$

□

Lemma 26. *Let m be a positive integer, $\beta \in \mathbb{R}$, and let $f(x)$ be an entire function. Then*

$$\delta_\beta(\delta_\beta - 1)(\delta_\beta - 2)\cdots\big(\delta_\beta - (m-1)\big)[f(x)] = (x - \beta D)^m D^m[f(x)] \tag{3.5}$$

where $\delta_\beta = xD - \beta D^2$.

Proof. We shall prove this lemma by induction on the positive integer m. For $m = 1$ equation (3.5) reduces to

$$\delta_\beta[f(x)] = (x - \beta D)D[f(x)],$$

which, by the definition of δ_β, is clearly true.

Now suppose that equation (3.5) holds for some given positive integer m. We first note that, by taking $r = m$ in Lemma 25,

$$\delta_\beta D^m[f(x)] = D^m(\delta_\beta - m)[f(x)].$$

Therefore

$$\begin{aligned}(x-\beta D)^{m+1}D^{m+1}[f(x)] &= (x-\beta D)^m \delta_\beta D^m[f(x)] \\ &= (x-\beta D)^m D^m(\delta_\beta - m)[f(x)] \\ &= \delta_\beta(\delta_\beta - 1)(\delta_\beta - 2)\cdots(\delta_\beta - m)[f(x)].\end{aligned}$$

Thus the Lemma holds for the integer $m+1$ as well. □

Lemma 27. *Suppose $m \geq 1$ and $p \geq 0$ are integers, $\beta \in \mathbb{R}$, and let $f(x)$ be an entire function. Then*

$$\begin{aligned}&\delta_\beta(\delta_\beta - 1)(\delta_\beta - 2)\cdots\big(\delta_\beta - (m-1)\big)\prod_{i=1}^{p}(\delta_\beta - b_i)[f(x)] \\ &\qquad\qquad = (x-\beta D)^m\left(\prod_{i=1}^{p}\big((m-b_i) + xD - \beta D^2\big)\right)D^m[f(x)],\end{aligned}$$

where $\delta_\beta = xD - \beta D^2$.

Proof. This is an immediate consequence of equations (3.5) and (3.4) of Lemmas 26 and 25, respectively. □

Again, it should be emphasized that the preceding technical lemmas are not that remarkable in themselves. We have only included them here due to the fact that they will be of use to us in what follows.

We will now include another result along these lines which *is* of some significance in itself. Indeed, the next lemma will be used in one of the major theorems of this dissertation (Theorem 147). Furthermore, it is the result contained in this lemma which led us to the consideration of the generalized Hermite polynomials $\mathcal{H}^{\alpha}$ defined by the generating relation (2.11) of the previous chapter.

Lemma 28. *For any $\alpha \in \mathbb{R}$ and any entire function $f(x)$,*

$$(x-\alpha D)^k[f(x)] = \sum_{j=0}^{k} \binom{k}{j} (-\alpha)^j \mathcal{H}_{k-j}^{(\alpha)}(x) f^{(j)}(x) \qquad (k=0,1,2,\ldots), \tag{3.6}$$

where D denotes differentiation with respect to x and $\mathcal{H}_n^{(\alpha)}(x)$ denotes the n^{th} generalized Hermite polynomial defined by the generating relation (2.11).

Proof. If $\alpha = 0$, then equation (3.6) reduces to

$$x^k f(x) = H_k^{(0)}(x) f(x),$$

which is true since $H_k^{(0)}(x) = x^k$.

We will prove that the lemma is true for $\alpha \neq 0$ by mathematical induction. For ease of notation, the superscript (α) of $\mathcal{H}_n^{(\alpha)}(x)$ will be suppressed.

For $k = 0$, equation (3.6) reduces to

$$f(x) = \mathcal{H}_0(x) f(x),$$

which is true since $\mathcal{H}_0(x) = 1$. Suppose that equation (3.6) holds for some given integer $k \geq 0$. Then

$$\begin{aligned}
(x-\alpha D)^{k+1}[f(x)] &= (x-\alpha D)\left[(x-\alpha D)^k[f(x)]\right] \\
&= (x-\alpha D)\left[\sum_{j=0}^{k} \binom{k}{j} (-\alpha)^j \mathcal{H}_{k-j}(x) f^{(j)}(x)\right],
\end{aligned}$$

which, by the product rule for differentiation, becomes

$$(x-\alpha D)^{k+1}[f(x)] = \sum_{j=0}^{k}\binom{k}{j}(-\alpha)^j x\mathcal{H}_{k-j}(x)f^{(j)}(x) + \sum_{j=0}^{k}\binom{k}{j}(-\alpha)^{j+1}\left[\mathcal{H}'_{k-j}(x)f^{(j)}(x)+\mathcal{H}_{k-j}(x)f^{(j+1)}(x)\right].$$

Gathering the derivatives of $f(x)$ of the same order and re-indexing, we have

$$(x-\alpha D)^{k+1}[f(x)] = \sum_{j=0}^{k}\binom{k}{j}(-\alpha)^j\left[x\mathcal{H}_{k-j}(x)-\alpha\mathcal{H}'_{k-j}(x)\right]f^{(j)}(x) + \sum_{j=1}^{k+1}\binom{k}{j-1}(-\alpha)^j\mathcal{H}_{k+1-j}(x)f^{(j)}(x). \tag{3.7}$$

Combining the pure recurrence relation satisfied by the generalized Hermite polynomials (2.15) with the differential recurrence relation (2.16), we obtain

$$\mathcal{H}_{n+1}(x) = x\mathcal{H}_n(x) - \alpha n\mathcal{H}_{n-1}(x) = x\mathcal{H}_n(x) - \alpha\mathcal{H}'_n(x),$$

which holds for all integers $n \geq 1$. Thus equation (3.7) becomes

$$(x-\alpha D)^{k+1}[f(x)] = \sum_{j=0}^{k}\binom{k}{j}(-\alpha)^j\mathcal{H}_{k+1-j}(x)f^{(j)}(x) + \sum_{j=1}^{k+1}\binom{k}{j-1}(-\alpha)^j\mathcal{H}_{k+1-j}(x)f^{(j)}(x). \tag{3.8}$$

Since, for any $j = 1, 2, 3, \ldots k$,

$$\binom{k}{j}+\binom{k}{j-1} = \frac{k!}{j!(k-j)!}+\frac{k!}{(j-1)!(k-j+1)!} = \frac{k!(k+1)}{j!(k+1-j)!} = \binom{k+1}{j},$$

we may rewrite equation (3.8) as

$$(x-\alpha D)^{k+1}[f(x)] = \sum_{j=0}^{k+1}\binom{k+1}{j}(-\alpha)^j\mathcal{H}_{k+1-j}(x)f^{(j)}(x)$$

as desired. Therefore equation (3.6) holds for every integer k. □

Let us now restrict our attention to linear operators on the vector space of real polynomials. There are several different ways in which one can define such an operator. Indeed, given any basis $Q = \{q_k(x)\}_{k=0}^{\infty}$ for $\mathbb{R}[x]$, we may define a linear operator T by its action on the basis elements $q_k(x)$. For example, given a sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$, we can define a linear operator T by

$$T[x^n] = \gamma_n x^n \qquad (n = 0, 1, 2, \dots),$$

and, by linearity, we have

$$T[a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n] = \gamma_0 a_0 + \gamma_1 a_1 x + \gamma_2 a_2 x^2 + \cdots + \gamma_n a_n x^n.$$

Similarly, for the same sequence, we could define a linear operator T_H on $\mathbb{R}[x]$ by

$$T_H[H_n(x)] = \gamma_n H_n(x) \qquad (n = 0, 1, 2, \dots),$$

where $H_n(x)$ denotes the n^{th} Hermite polynomial. Operators of this form, which take basis elements into scalar multiples of themselves, will play a significant role in our investigation.

It should also be noted that a linear operator on $\mathbb{R}[x]$ is uniquely determined by its action on basis elements. Thus, if two operators T_1 and T_2 agree at each element of some basis for $\mathbb{R}[x]$, then $T_1 = T_2$. This useful fact will often be used when showing that two linear operators on $\mathbb{R}[x]$ are equal.

3.2 Representation as Differential Operators

There are many different ways to define a linear operator on the vector space of complex polynomials. In the midst of such variety, it is a remarkable fact that, no matter how a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$ is defined, it can always be represented formally as a differential operator with complex polynomial coefficients. The following proposition is known but, in the absence of a good reference, we will provide its proof for the sake of completeness.

Proposition 29. *Let $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator. Then there exists a unique set of complex polynomials $\{p_k(x)\}_{k=0}^{\infty}$ such that*

$$T[f(x)] = \sum_{k=0}^{\infty} p_k(x) f^{(k)}(x)$$

for all $f(x) \in \mathbb{C}[x]$.

Proof. Let T be a linear operator on the set of complex polynomials. Define the polynomials $\{p_k(x)\}_{k=0}^{\infty}$ recursively by

$$p_0(x) = T[1]$$

and

$$p_n(x) = \frac{1}{n!}\left(T\left[x^n\right] - \sum_{k=0}^{n-1} p_k(x) D^k x^n\right) \qquad (n = 1, 2, 3, \dots), \tag{3.9}$$

where D denotes differentiation with respect to x.

Suppose $f(x) = \sum_{k=0}^{n} a_k x^k$ is a complex polynomial. Then, by the linearity of T,

$$T[f(x)] = T\left[\sum_{k=0}^{n} a_k x^k\right] = \sum_{k=0}^{n} a_k T\left[x^k\right]. \tag{3.10}$$

Furthermore, by equation (3.9),

$$T\left[x^k\right] = \sum_{j=0}^{k} p_j(x) D^j x^k. \tag{3.11}$$

Combining equations (3.10) and (3.11) yields

$$T[f(x)] = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} p_j(x) D^j x^k = \sum_{k=0}^{n} \sum_{j=0}^{k} a_k p_j(x) D^j x^k. \tag{3.12}$$

Since $D^j x^k = 0$ for $j > k$, we may write equation (3.12) as

$$T[f(x)] = \sum_{k=0}^{n} \sum_{j=0}^{n} a_k p_j(x) D^j x^k.$$

Rewriting this double sum yields

$$\begin{aligned} T[f(x)] &= \sum_{k=0}^{n} \sum_{j=0}^{n} a_k p_j(x) D^j x^k \\ &= \sum_{j=0}^{n} \sum_{k=0}^{n} a_k p_j(x) D^j x^k \\ &= \sum_{j=0}^{n} p_j(x) D^j \sum_{k=0}^{n} a_k x^k \\ &= \sum_{j=0}^{n} p_j(x) f^{(j)}(x) \end{aligned}$$

as desired.

To show that this representation is unique, suppose there exists another set of complex polynomials $\{q_k(x)\}_{k=0}^{\infty}$ such that

$$T[f(x)] = \sum_{k=0}^{\infty} q_k(x) f^{(k)}(x)$$

for all $f(x) \in \mathbb{C}[x]$. Then, in particular,

$$\sum_{k=0}^{n} p_k(x) D^k x^n = T\left[x^n\right] = \sum_{k=0}^{n} q_k(x) D^k x^n \qquad (n = 0, 1, 2, \dots). \tag{3.13}$$

For $n = 0$, equation (3.13) becomes $p_0(x) = q_0(x)$. When $n = 1$, equation (3.13) becomes

$$p_0(x)x + p_1(x) = q_0(x)x + q_1(x),$$

whence $p_1(x) = q_1(x)$. Now suppose that there exists an integer $n \geq 1$ such that $p_k(x) = q_k(x)$ for all $k \in \{1, 2, \ldots, n\}$. Then, since

$$\sum_{k=0}^{n+1} p_k(x) D^k x^{n+1} = \sum_{k=0}^{n+1} q_k(x) D^k x^{n+1},$$

it follows that $p_{n+1}(x) = q_{n+1}(x)$. Therefore, by the principle of strong induction, $p_n(x) = q_n(x)$ for all integers n. □

Example 30. The linear operator T defined by $T[x^n] = \mathcal{H}_n^{(\alpha)}(x)$, where $\mathcal{H}_n^{(\alpha)}(x)$ is the n^{th} generalized Hermite polynomial with real parameter α (2.14), has the representation

$$T = \exp\left(-\frac{\alpha}{2} D^2\right) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{2^k k!} D^{2k}.$$

To see this, we note that, for any non-negative integer n,

$$\exp\left(-\frac{\alpha}{2} D^2\right)[x^n] = \left(\sum_{k=0}^{\infty} \frac{(-\alpha)^k}{2^k k!} D^{2k}\right)[x^n] = \sum_{k=0}^{[n/2]} \frac{(-\alpha)^k n! x^{n-2k}}{2^k k! (n-2k)!} = \mathcal{H}_n^{(\alpha)}(x). \quad (3.14)$$

Incidentally, from the relation between the generalized Hermite and classical Hermite polynomials (2.13), we obtain the formula

$$\exp\left(-\frac{\alpha}{2} D^2\right)[x^n] = \left(\frac{\alpha}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2\alpha}}\right) \qquad (\alpha \neq 0;\ n = 0, 1, 2, \ldots), \qquad (3.15)$$

which is used in various forms by several authors (see, e.g., [1, p. 184], [7, p. 181], [15, p. 432], [16, p. 564], and [18, p. 377]).

Example 31. The linear operator T defined by $T[x^n] = nx^n$ can be represented as $T = xD$. Indeed,

$$(xD)[x^n] = nx^n \qquad (n = 0, 1, 2, \ldots).$$

Example 32. The linear operator T defined by $T[x^n] = (1 + n + n^2)x^n$ can be represented as $T = 1 + 2xD + x^2D^2$. Indeed,

$$\big(1 + 2xD + D^2\big)[x^n] = \big(1 + 2n + n(n-1)\big)x^n = (1 + n + n^2)x^n \qquad (n = 0, 1, 2, \dots).$$

In general, the differential operator representation of a linear operator which corresponds to a real sequence $\{\gamma_k\}_{k=0}^{\infty}$ has a beautiful representation in terms of the "reverse" of the Jensen polynomials associated with the sequence.

Proposition 33. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let*

$$g_n^*(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^{n-k} \qquad (n = 0, 1, 2, \dots).$$

Then the linear operator T on the set of real (or complex) polynomials defined by $T[x^n] = \gamma_n x^n$ $(n = 0, 1, 2, \dots)$ can be represented as

$$T = \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k D^k, \tag{3.16}$$

where D denotes differentiation with respect to x.

Proof. Let

$$\widehat{T} = \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k D^k \qquad \left(D = \frac{d}{dx}\right)$$

be the differential operator which appears in equation (3.16). To show that $\widehat{T} = T$, it suffices to show that

$$\widehat{T}[x^n] = \gamma_n x^n \qquad (n = 0, 1, 2, \dots). \tag{3.17}$$

For any integer $n \geq 0$,

$$\widehat{T}[x^n] = \left(\sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k D^k \right) [x^n] = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{g_k^*(-1)}{k!} x^n = \sum_{k=0}^{n} \binom{n}{k} g_k^*(-1) x^n. \tag{3.18}$$

If we set

$$s_n = \sum_{k=0}^{n} \binom{n}{k} g_k^*(-1) \qquad (n = 0, 1, 2, \dots),$$

then, to prove that $\widehat{T} = T$, it suffices to show (compare (3.17) and (3.18)) that

$$s_n = \gamma_n \qquad (n = 0, 1, 2, \dots).$$

First note that, by the definition of s_n and $g_k^*(x)$,

$$s_n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \gamma_j = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} \gamma_j. \tag{3.19}$$

Changing the order of summation in equation (3.19) and rewriting yields

$$s_n = \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^{k-j} \gamma_j = \sum_{j=0}^{n} \frac{\gamma_j}{j!} \sum_{k=j}^{n} \binom{n}{k} \frac{k!}{(k-j)!} (-1)^{k-j} \tag{3.20}$$

Noting that

$$\frac{d^j}{dx^j}(1+x)^n = \sum_{k=j}^{n} \binom{n}{k} \frac{k!}{(k-j)!} x^{k-j},$$

equation (3.20) can be rewritten and simplified to

$$s_n = \sum_{j=0}^{n} \frac{\gamma_j}{j!} \left[\frac{d^j}{dx^j}(1+x)^n \right]_{x=-1} = \gamma_n$$

as desired. □

Remark 34. It is worthy to note that, in the extensive literature that deals with linear operators which are defined by $T[x^n] = \gamma_n x^n$ for some real sequence $\{\gamma_k\}_{k=0}^{\infty}$, the representation of these operators given in Proposition 33 appears to be new.

Remark 35. Proposition 33 could be obtained using other methods. Indeed, one could use the relation

$$\gamma_n x^n = T[x^n] = \sum_{k=0}^{n} T_k(x) \frac{n!}{(n-k)!} x^{n-k} \qquad (n = 0, 1, 2, \dots)$$

and an induction argument to prove Proposition 33. Yet another method employs a matrix representation. If we identify each real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ with the sequence which arises from its coefficients

$$\big(a_0,\, a_1,\, a_2, \dots,\, a_n,\, 0,\, 0,\, 0,\, \dots\big),$$

then we can represent any linear operator T on $\mathbb{R}[x]$ by an infinite-dimensional matrix M_T which has the sequence corresponding to $T[x^n]$ as its n^{th} column. For example, the differentiation operator D is represented by the matrix

$$M_D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4 & \\ 0 & 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

Indeed,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 5 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ 0 \\ \vdots \end{pmatrix} = \big(a_1,\, 2a_2,\, 3a_3,\, 4a_4,\, \dots,\, na_n,\, 0,\, 0,\, \dots\big).$$

Now, if we let

$$T_n(x) = \sum_{k=0}^{\infty} a_{nk} x^k \qquad (n = 0, 1, 2, , \dots)$$

be a sequence of real polynomials (i.e., only finitely many of the coefficients in T_n are non-zero), then we may represent the operator $T = \sum_{k=0}^{\infty} T_k(x)D^k$ by the following matrix.

$$M_T = \begin{pmatrix} a_{00} & a_{10} & 2!a_{20} & 3!a_{30} & \dots \\ a_{01} & a_{00}+a_{11} & 2a_{10}+2!a_{21} & (3\cdot 2)a_{20}+3!a_{31} & \dots \\ a_{02} & a_{01}+a_{12} & a_{00}+2a_{11}+2!a_{22} & 3a_{10}+(3\cdot 2)a_{21}+3!a_{32} & \dots \\ a_{03} & a_{02}+a_{13} & a_{01}+2a_{12}+2!a_{23} & a_{00}+3a_{11}+(3\cdot 2)a_{22}+3!a_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If the operator T is given by $T[x^n] = \gamma_n x^n$, for some real sequence $\{\gamma_k\}_{k=0}^{\infty}$, then the matrix representation must be a diagonal matrix with γ_k on the diagonal of the k^{th} row (for this reason, linear operators arising in this way are sometimes referred to as *diagonal operators*). Comparing with the matrix representation M_T, it is easy to see that $a_{ij} = 0$ whenever $i \neq j$. Thus

$$T_n(x) = a_{nn}x^n \qquad (n = 0, 1, 2, \dots)$$

and

$$\gamma_n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} a_{kk} \qquad (n = 0, 1, 2, \dots)$$

from which Proposition 33 could be proved via an induction argument.

Neither the induction argument, nor the matrix representation, seem to yield a proof which is more elegant than the one that was given. However, we mention these methods due to the fact that they may help shed light on another question. Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and define the linear operator T_H by

$T_H[H_n(x)] = \gamma_n H_n(x)$, where H_n is the n^{th} Hermite polynomial. By Proposition 29, this operator may be represented in the form

$$T_H = \sum_{k=0}^{\infty} T_k(x) D^k \qquad \big(T_k \in \mathbb{R}[x]\big).$$

The question is whether or not there is an explicit formula which defines the polynomials T_k in this representation. If such a formula exists, the formula for the product of two Hermite polynomials (2.10) is undoubtedly relevant.

3.3 Linear Operators Which Preserve Reality of Zeros

Although the following terminology is not standard, we shall use it due to its intuitive nature.

Definition 36. A linear operator $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ is said to *preserve reality of zeros* if it has the property that

$$T[p(x)] \in \mathcal{L} - \mathcal{P} \qquad \text{whenever} \qquad p(x) \in (\mathbb{R}[x] \cap \mathcal{L} - \mathcal{P})\,. \tag{3.21}$$

Thus, T preserves reality of zeros if and only if T maps polynomials with only real zeros to polynomials with only real zeros or, perhaps, to the identically zero function. For example, as a consequence of Rolle's theorem, the differentiation operator $D = \dfrac{d}{dx}$ preserves reality of zeros. More generally, if $p(x) \in \mathbb{R}[x]$ has only real zeros, then the operator $p(D)$ preserves reality of zeros, which is a consequence of the well-known Hermite-Poulain Theorem.

Theorem 37. (Hermite-Poulain Theorem [20, p. 337], [23, p. 4]) *Let*

$$h(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

be a real polynomial with only real zeros. Then, for any real polynomial $f(x)$, the number of non-real zeros of

$$h(D)f(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x)$$

does not exceed the number of non-real zeros of $f(x)$.

The most elegant proof of this theorem is again due to Rolle's theorem. Indeed, the operator $h(D)$ can be factored into operators of the form $\alpha + D$ and

$$\frac{d}{dx}\big(e^{\alpha x} f(x)\big) = \big(\alpha f(x) + f'(x)\big)e^{\alpha x}.$$

As the next theorem demonstrates, one can actually take $h(x)$ in the Hermite-Poulain Theorem to be a transcendental function in the class $\mathcal{L} - \mathcal{P}$. This important fact will be useful to us in our later investigations.

Theorem 38. *Let*

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{L} - \mathcal{P}.$$

Then, for any real polynomial $f(x)$, the number of non-real zeros of

$$\varphi(D)[f(x)] = \sum_{k=0}^{\infty} a_k f^{(k)}(x)$$

does not exceed the number of non-real zeros of $f(x)$.

Proof. Since $\varphi(x) \in \mathcal{L} - \mathcal{P}$, there is a sequence of polynomials

$$p_m(x) = \sum_{k=0}^{n_m} a_{m,k} x^k \qquad (m = 1, 2, 3, \dots),$$

each of which has only real zeros, which converge uniformly on compact subsets of $\mathbb{C}$ to $\varphi(x)$. The coefficients of $p_m(x)$ tend to those of $\varphi(x)$, i.e.,

$$\lim_{m\to\infty} a_{m,k} = a_k \qquad (k = 0, 1, 2, \dots).$$

Thus, for any real polynomial $f(x)$, the sequence of polynomials $\{p_m(D)[f(x)]\}_{m=1}^{\infty}$ converge uniformly on compact subsets to $\varphi(D)[f(x)]$. Therefore, by the Hermite-Poulain Theorem and Hurwitz' theorem (see Corollary 12), the number of non-real zeros of $\varphi(D)[f(x)]$ does not exceed the number of non-real zeros of $f(x)$. $\square$

One of the fundamental results which gives rise to linear operators which preserve reality of zeros is is the following composition theorem.

Theorem 39. (Schur-Maló Composition Theorem [12, p. 7],[20, pp. 337-340]) *Suppose that all the zeros of the polynomial* $p(x) = \sum_{k=0}^{m} a_k x^k$ $(a_m \neq 0)$ *are real and all zeros of the polynomial* $q(x) = \sum_{k=0}^{n} b_k x^k$ $(b_n \neq 0)$ *are real and of the same sign. If we set* $\nu = \min\{m, n\}$, *then all of the zeros of the polynomials* $f(x) = \sum_{k=0}^{\nu} k! a_k b_k x^k$ *and* $g(x) = \sum_{k=0}^{\nu} a_k b_k x^k$ *are also real.*

The composition theorem provides a powerful tool in studying the diagonal operators, i.e., operators defined by $T[x^n] = \gamma_n x^n$ for some real sequence $\{\gamma_k\}_{k=0}^{\infty}$, which preserve reality of zeros. The sequences which give rise to these operators are called *multiplier sequences*. To be precise, we have the following definition.

Definition 40. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier sequence* if the corresponding linear operator T, defined by $T[x^n] = \gamma_n x^n$, has the property that

$$T[p(x)] \in \mathcal{L}-\mathcal{P} \qquad \text{whenever} \qquad p(x) \in (\mathbb{R}[x] \cap \mathcal{L}-\mathcal{P}).$$

Let us begin the discussion of multiplier sequences with some introductory examples.

Example 41. Fix a non-zero real number r and define the linear operator T by $T[x^n] = r^n x^n$. If $p(x)$ is a real polynomial having only real zeros, then $T[p(x)] = p(rx)$ also has only real zeros. Thus, for any non-zero real number r, the geometric sequence $\{r^k\}_{k=0}^{\infty}$ is a multiplier sequence.

Example 42. Another elementary example of a multiplier sequence is the sequence $\{k\}_{k=0}^{\infty}$. To see this, we only need to note that, for every non-negative integer n, $(xD)[x^n] = nx^n$, and the operator xD preserves reality of zeros.

In order to demonstrate a more interesting example of a multiplier sequence, we prove the following lemma.

Lemma 43. *Suppose that the complex polynomial*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad (a_n \neq 0)$$

has only real zeros. Then the complex polynomial

$$f^*(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

also has only real zeros.

Proof. The lemma is a consequence of the relation

$$f^*(x) = x^n f\left(\frac{1}{x}\right).$$

□

Example 44. The sequence $\left\{\dfrac{1}{k!}\right\}_{k=0}^{\infty}$ is a multiplier sequence. Indeed, if $p(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, then the same is true of the polynomial $p^*(x) = \sum_{k=0}^{n} a_k x^{n-k}$. Thus, by the Hermite-Poulain Theorem,

$$p^*(D)\left[\frac{x^n}{n!}\right] = \sum_{k=0}^{n} \frac{a_k}{k!} x^k$$

has only real zeros.

Before citing more examples, we will give several properties of multiplier sequences which are readily verified.

Proposition 45. ([9], [20, p. 341]) *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence. Then*

1. *The sequence $\{\gamma_k\}_{k=m}^{\infty}$ is also a multiplier sequence, where m is any non-negative integer.*

2. *If there exists an integer $m \geq 0$ such that $\gamma_m \neq 0$ and an integer $n > m$ such that $\gamma_n = 0$, then $\gamma_k = 0$ for all $k \geq n$.*

3. *The elements of $\{\gamma_k\}_{k=0}^{\infty}$ are either all of the same sign, or they alternate in sign.*

4. *The sequence $\left\{(-1)^k \gamma_k\right\}_{k=0}^{\infty}$ is also a multiplier sequence.*

5. *For any $r \in \mathbb{R}$, the sequence $\{r\gamma_k\}_{k=0}^{\infty}$ is also a multiplier sequence.*

6. *The elements of $\{\gamma_k\}_{k=0}^{\infty}$ satisfy Turán's inequality*

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \qquad (k = 1, 2, 3, \ldots).$$

In 1883, Laguerre [19] proved that the sequences

$$\left\{1, \frac{1}{\alpha+\omega}, \frac{1}{(\alpha+\omega)(2\alpha+\omega)}, \frac{1}{(\alpha+\omega)(2\alpha+\omega)(3\alpha+\omega)}, \dots\right\} \qquad (\alpha, \omega > 0)$$

and

$$\left\{1, q, q^4, q^9, \dots, q^{n^2}, \dots\right\} \qquad (-1 \leq q \leq 1)$$

are multiplier sequences. In 1911, Jensen $\big($[17], [20, p. 343]$\big)$ invoked the Schur-Maló Composition Theorem to show that, for any positive integer n, the sequence

$$\left\{1, 1, \left(1-\frac{1}{n}\right), \dots, \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right), 0, 0, 0, \dots\right\}$$

is a multiplier sequence. In 1914, Pólya and Schur completely characterized multiplier sequences as follows.

Theorem 46. (Pólya-Schur [26], [20, Chapter VIII], [23, Kapitel II]) *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of non-negative real numbers and let* T *be the linear operator on* $\mathbb{R}[x]$ *defined by* $T[x^n] = \gamma_n x^n$*. Then the following are equivalent.*

1. $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence.*
2. (*Transcendental Characterization*)

$$T[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}-\mathcal{P}^+.$$

3. (*Algebraic Characterization*) *For each* $n = 0, 1, 2, \dots$

$$T[(1+x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{L}-\mathcal{P}^+.$$

Although the preceding theorem only applies to non-negative sequences, Properties 3-5 of Proposition 45 complete the characterization of multiplier sequences.

Until now, all linear operators have been only defined on vector spaces which consist of polynomials. However, the Transcendental Characterization of multiplier sequences in Theorem 46 requires that we be able to apply the multiplier sequence to the Taylor coefficients of the *transcendental* entire function e^x. The next theorem shows that it does, in fact, make sense to apply a multiplier sequence to any function in the Laguerre-Pólya class.

Theorem 47. ([20, p. 343]) *Suppose* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence and let* T *be the corresponding operator defined by* $T[x^n] = \gamma_n x^k$. *If* $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{L}-\mathcal{P}$, *then the function* $T[f(x)] = \sum_{k=0}^{\infty} a_k \gamma_k x^k$ *represents an entire function, and this entire function also belongs to the Laguerre-Pólya class.*

Let us now employ Pólya and Schur's characterization to give an example of a multiplier sequence which will be of great interest to us later on.

Example 48. The sequence $\left\{1+k+k^2\right\}_{k=0}^{\infty}$ is a multiplier sequence. Let us first show this using the algebraic characterization of multiplier sequences given in Theorem 46. Let T be the linear operator defined by $T[x^n] = (1+n+n^2)x^n$. Then the differential operator representation of T is $T = 1 + 2xD + D^2$, which follows from the relation

$$\left(1 + 2xD + x^2D^2\right)[x^n] = (1+n+n^2)x^n \qquad (n = 0, 1, 2, \ldots).$$

Thus,

$$\begin{aligned} T\big[(x+1)^n\big] &= \big(1+2xD+x^2D^2\big)\big[(1+x)^n\big] \\ &= (x+1)^{n-2}\big((n^2+n+1)x^2+2(n+1)x+1\big). \end{aligned} \tag{3.22}$$

And, since the discriminant of the quadratic polynomial in equation (3.22) is $4n \geq 0$, $T\big[(x+1)^n\big]$ has only real zeros for any integer $n \geq 0$. Therefore, the sequence $\big\{1+k+k^2\big\}_{k=0}^{\infty}$ is a multiplier sequence.

Alternatively, we could use the transcendental characterization of multiplier sequences given in Theorem 46. Since

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{1+k+k^2}{k!} x^k = (x+1)^2 e^x \in \mathcal{L}-\mathcal{P}^+,$$

the sequence $\big\{1+k+k^2\big\}_{k=0}^{\infty}$ is a multiplier sequence.

To cite a result regarding linear operators which arise from applying a given real sequences to a basis *other than* the standard basis $\big\{x^k\big\}_{k=0}^{\infty}$, we have the following classical result which Turán announced in 1950.

Theorem 49. (Turán [29, p. 289], [1, p. 178]) *Suppose the real polynomial* $\sum_{k=0}^{n} a_k H_k(x)$, *where* H_k *denotes the* k^{th} *Hermite polynomial, has only real zeros. If* $g(x)$ *is a polynomial having only real negative zeros, then the polynomial* $\sum_{k=0}^{n} a_k g(k) H_k(x)$ *also has only real zeros.*

In 2001 Bleecker and Csordas [1, Theorem 2.7] proved and generalized this result. They demonstrated that one can take $g(x)$ to be any transcendental function in the class $\mathcal{L}-\mathcal{P}^+$. Their investigations led them to state the following open problem.

Problem 50. (Bleecker-Csordas [1, Problem 4.1]) Characterize all real sequences $\{\gamma_k\}_{k=0}^{\infty}$ such that

$$\text{if } \sum_{k=0}^{n} a_k H_k(x) \in \mathcal{L}-\mathcal{P}, \text{ then } \sum_{k=0}^{n} \gamma_k a_k H_k(x) \in \mathcal{L}-\mathcal{P}, \tag{3.23}$$

where H_k denotes the k^{th} Hermite polynomial.

Remark 51. Problem 50 was the source of inspiration for a large portion of the research contained in this dissertation, and we shall see its complete solution in the chapters which follow.

It is worthy to note that Bleecker and Csordas were able to generalize Turán's theorem by discovering another linear operator on $\mathbb{R}[x]$ which preserves reality of zeros.

Proposition 52. (Bleecker-Csordas [1, Lemma 2.2]) *Suppose that the real polynomial $p(x)$ has only real zeros. Then, for any fixed $\alpha \geq 0$ and $\beta \geq 0$,*

$$f(x) = \alpha p(x) + xp'(x) - \beta p''(x) \in \mathcal{L}-\mathcal{P}.$$

Remark 53. Proposition 52 states that, for any non-negative constants α and β, the operator $\alpha + xD - \beta D^2$ preserves reality of zeros. Operators of this form arise naturally in connection with the Hermite polynomials. Indeed, since the Hermite polynomials satisfy Hermite's differential equation (2.7) we have

$$\left(xD - \frac{1}{2}D^2\right)[H_n(x)] = nH_n(x) \qquad (n = 0, 1, 2, \ldots). \tag{3.24}$$

By iterating relation (3.24) we see that, for any entire function $g(x)$,

$$g\left(xD - \frac{1}{2}D^2\right)[H_n(x)] = g(n)H_n(x) \qquad (n = 0, 1, 2, \dots). \tag{3.25}$$

Thus, Turán's theorem (Theorem 49) follows immediately from relation (3.25) and Proposition 52.

In the next section we will show that operators of the form $\alpha + xD - \beta D^2$, with $\alpha, \beta \geq 0$, satisfy another property which is stronger than the property of preserving reality of zeros. We will also show that, under certain restrictions on the degree of the polynomial $p(x)$, we may replace the term $xp'(x)$ of $f(x)$ in Proposition 52 by the term $(cx + d)p'(x)$, where c and d are real constants. These generalizations will provide a way to further generalize Turán's theorem (Theorem 49).

To conclude this section, we mention a very recent result which completely characterizes linear operators which preserve reality of zeros in terms of the distribution of zeros of certain functions in two variables. Given a linear operator $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$, we extend the operator to the vector space $\mathbb{R}[x, y]$ by declaring $T[x^n y^m] = y^m T[x^n]$ for all non-negative integers n and m. Thus the extension is obtained by essentially treating the second variable as a scalar. In 2006, Borcea, Brändén and Shapiro [2] used this extension to establish an algebraic characterization of linear operators which preserve reality of zeros.

Theorem 54. (Borcea, Brändén and Shapiro [2, Corollary 1]) *A linear operator $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ preserves reality of zeros if and only if either*

(a) *T has range of dimension no greater than two and is of the form*

$$T[f(x)] = \alpha(f(x))P(x) + \beta(f(x))Q(x),$$

where α and β are linear functionals on $\mathbb{R}[x]$ and $P(x)$ and $Q(x)$ are real polynomials which have only real zeros which interlace, or

(b) *Each of the polynomials in one of the sets $\{T[(x+y)^n]\}_{n=0}^{\infty}$ or $\{T[(x-y)^n]\}_{n=0}^{\infty}$ do not have any zeros in the set $\mathbf{H} = \{(x,y) \in \mathbb{C}^2 : \mathrm{Im}(x) > 0,\, \mathrm{Im}(y) > 0\}$.*

In the same paper, Borcea, Brändén and Shapiro also gave the following transcendental characterization of linear operators which preserve reality of zeros.

Theorem 55. (Borcea, Brändén and Shapiro [2, Theorem 5]) *A linear operator $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ preserves reality of zeros if and only if either*

(a) *T has range of dimension no greater than two and is of the form*

$$T[f(x)] = \alpha(f(x))P(x) + \beta(f(x))Q(x),$$

where α and β are linear functionals on $\mathbb{R}[x]$ and $P(x)$ and $Q(x)$ are real polynomials which have only real zeros which interlace, or

(b) *One of the expressions $\displaystyle\sum_{k=0}^{\infty} \frac{(-y)^k}{k!} T[x^k]$ or $\displaystyle\sum_{k=0}^{\infty} \frac{(-y)^k}{k!} T[-x^k]$ represents an entire function in two variables which is the uniform limit on compact subsets of polynomials which do not have any zeros in the set*

$$\mathbf{H} = \{(x,y) \in \mathbb{C}^2 : \mathrm{Im}(x) > 0,\, \mathrm{Im}(y) > 0\}. \tag{3.26}$$

While these characterizations are quite amazing, they can be difficult to apply in practice. In general, it seems to be rather difficult to determine whether or not a polynomial in two variables has all its zeros outside of the set (3.26).

3.4 Complex Zero Decreasing Operators

In what follows, we will often be counting non-real zeros of a given function. To facilitate the discussion, we will adopt the following notation.

Notation 56. For any entire function $f(x)$, which is not identically zero, let $Z_C\big(f(x)\big)$ denote the number of non-real zeros of $f(x)$, counting multiplicities. For convenience, we shall also define $Z_C(0) = 0$.

Again, the following terminology is not standard, but we shall use it due to its intuitive nature.

Definition 57. A linear operator $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ is called a *complex zero decreasing operator* if it has the property that, for every real polynomial $p(x)$,

$$Z_C\big(T[p(x)]\big) \leq Z_C\big(p(x)\big). \tag{3.27}$$

Thus, T is a complex zero decreasing operator if and only if T does not increase the number of non-real zeros of any real polynomial (except, possibly, for real polynomials which it takes to the identically zero function). In particular, every complex zero decreasing operator must also preserve reality of zeros.

We saw in the previous section that the differentiation operator $D = \dfrac{d}{dx}$ preserves reality of zeros. In fact, the differentiation operator is a complex zero decreasing operator, which follows from Rolle's theorem. More generally, by the Hermite-Poulain Theorem (Theorem 37), if $p(x)$ is a real polynomial with only real zeros, then $p(D)$ is a complex zero decreasing operator.

The subclass of diagonal complex zero decreasing operators, which are defined by $T[x^n] = \gamma_n x^n$ for some real sequence $\{\gamma_k\}_{k=0}^{\infty}$, have been studied by several authors. In analogy to multiplier sequences, the following definition is commonly used.

Definition 58. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called a *complex zero decreasing sequence*, or CZDS for brevity (which we will also use in the plural), if the corresponding linear operator T, defined by $T[x^n] = \gamma_n x^n$, has the property that, for every real polynomial $p(x)$,

$$Z_C\big(T[p(x)]\big) \leq Z_C\big(p(x)\big).$$

Remark 59. It easy to see that the sequences $\left\{r^k\right\}_{k=0}^{\infty}$ and $\{k\}_{k=0}^{\infty}$ of Examples 41 and 42, respectively, are CZDS. However, whether or not the sequence $\left\{\dfrac{1}{k!}\right\}_{k=0}^{\infty}$ of Example 44 is a CZDS is not so clear. Perhaps the Hermite-Poulain Theorem could be adapted to handle polynomials which do not necessarily have only real zeros. In particular, it would be desirable to know that, if the real polynomial p has $2d$ non-real zeros, then $p(D)$ will not increase the number of non-real zeros of any real polynomial by more than $2d$. However this is not the case. For example, if $p(x) = x^2+1$ and $q(x) = (x^2-4)(x^2-9)$. Then $Z_C\big(q(x)\big) = 0$, while $Z_C\big(p(D)[q(x)]\big) =$

$Z_C(x^4 - x^2 + 10) = 4$. Thus, the operator $p(D)$ has increased the number of non-real zeros of q by four, while $p(x)$ only has two non-real zeros. As it turns out, the sequence $\left\{\frac{1}{k!}\right\}_{k=0}^{\infty}$ *is* a CZDS. This is a consequence of a theorem due to Laguerre which was subsequently extended by Pólya [24].

Theorem 60. (Laguerre's Theorem [23, p. 6], [12, p. 23])

1. *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n *and let* $h(x)$ *be a polynomial with only real zeros, none of which lie in the interval* $(0, n)$*. Then* $Z_C\left(\sum_{k=0}^{n} h(k) a_k x^k\right) \le Z_C\big(f(x)\big)$.

2. *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*, let* $\varphi(x) \in \mathcal{L}-\mathcal{P}$*, and suppose that none of the zeros of* φ *lie in the interval* $(0, n)$*. Then* $Z_C\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) \le Z_C\big(f(x)\big)$.

3. *If* $\varphi \in \mathcal{L}-\mathcal{P}(-\infty, 0]$*, then the sequence* $\{\varphi(k)\}_{k=0}^{\infty}$ *is a CZDS.*

Example 61. Since $\frac{1}{\Gamma(x+1)} \in \mathcal{L}-\mathcal{P}(-\infty, 0]$, where $\Gamma(x)$ denotes the Gamma function, the sequence

$$\left\{\frac{1}{\Gamma(k+1)}\right\}_{k=0}^{\infty} = \left\{\frac{1}{k!}\right\}_{k=0}^{\infty}$$

is a CZDS.

Let us now demonstrate that there are non-trivial CZDS which cannot be interpolated by functions in $\mathcal{L}-\mathcal{P}^{+}$.

Example 62. For any integer $m \geq 2$, the sequence $\{k(k-1)\cdots(k-m+1)\}_{k=0}^{\infty}$ is a CZDS. To see this, we only need to note that

$$\big(x^m D^m\big)[x^n] = n(n-1)\cdots(n-m+1)x^n \qquad (n = 0, 1, 2, \dots),$$

and the operator $x^n D^n$ is, by Rolle's theorem, a complex zero decreasing operator. If there were a function

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{L}-\mathcal{P}^{+} \qquad (a_k \geq 0;\ k = 0, 1, 2, \dots)$$

which interpolated the sequence $\{k(k-1)\cdots(k-m+1)\}_{k=0}^{\infty}$, then, in particular, we have

$$0 = \varphi(1) = \sum_{k=0}^{\infty} a_k \qquad (a_k \geq 0;\ k = 0, 1, 2, \dots).$$

Therefore, $a_k = 0$ for all k, but this contradicts the fact that $\varphi(m) = m! \neq 0$. Whence, for any $m \geq 2$, the CZDS $\{k(k-1)\cdots(k-m+1)\}_{k=0}^{\infty}$ cannot be interpolated by a function in $\mathcal{L}-\mathcal{P}^{+}$.

Part 1 of Laguerre's theorem asserts that any sequence which can be interpolated by a real polynomial which has only real negative zeros is a CZDS. It turns out that the converse is true under the additional hypothesis that the interpolating polynomial does not vanish at the origin. In the case where the interpolating polynomial does vanish at the origin, it turns out that the sequence must be of a special form related to Example 62. The following characterization of polynomials which interpolate CZDS was given by Craven and Csordas in 1991.

Theorem 63. (Craven-Csordas [11, p. 13]) *Let $h(x)$ be a real polynomial. Then $\{h(k)\}_{k=0}^{\infty}$ is a CZDS if and only if either*

1. *$h(0) \neq 0$ and $h(x)$ has only real negative zeros, or*

2. *$h(0) = 0$ and $h(x)$ is of the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{k=1}^{p}(x-b_k)$$

where $m \geq 1$ and $p \geq 0$ are integers and $b_k < m$ for $k = 1, 2, 3, \ldots, p$.

It is now easy to demonstrate the existence of a multiplier sequence (see Example 48) which is *not* a CZDS.

Example 64. The multiplier sequence $\left\{1 + k + k^2\right\}_{k=0}^{\infty}$ is not a CZDS. Indeed, this sequence can be interpolated by the polynomial $h(x) = 1 + x + x^2$ which has two non-real zeros. Therefore, by Theorem 63, this sequence is not a CZDS. Alternatively, if the linear operator T is defined by $T[x^n] = (1 + n + n^2)x^n$ then, as was noted in [11, p. 5], the polynomial

$$p(x) = (x+1)^6\left(x^2 + \frac{1}{2}x + \frac{1}{5}\right)$$

has two non-real zeros, while the polynomial

$$T[p(x)] = \frac{1}{10}(x+1)^4\left(730x^4 + 785x^3 + 306x^2 + 43x + 2\right)$$

has four non-real zeros. This concrete example demonstrates, again, that the sequence $\left\{1 + k + k^2\right\}_{k=0}^{\infty}$ is not a CZDS.

It should be noted that, in contrast to multiplier sequences, there is no complete characterization of CZDS. Indeed, the following tantalizing problem remains open.

Problem 65. ([12, p. 26]) Is $\left\{e^{-k^3}\right\}_{k=0}^{\infty}$ a CZDS?

In the previous section, Turan's theorem (Theorem 49) gave an example of linear operators, each of which preserves reality of zeros, which are defined by applying the elements of a real sequence to a basis *other than* the standard basis $\left\{x^k\right\}_{k=0}^{\infty}$. In general, there does not seem to be *any* known results in this direction for complex zero decreasing operators. Thus the following problem remains open for all choices of Q, and very little is known if Q is not the standard basis.

Problem 66. Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a basis for the vector space of real polynomials $\mathbb{R}[x]$. Can one characterize the real sequences $\{\gamma_k\}_{k=0}^{\infty}$ which have the property that, for any real polynomial $\sum_{k=0}^{n} a_k q_k(x)$,

$$Z_C\left(\sum_{k=0}^{n} a_k \gamma_k q_k(x)\right) \leq Z_C\left(\sum_{k=0}^{n} a_k q_k(x)\right)? \tag{3.28}$$

In order to demonstrate the existence of non-trivial sequences with this property for the Hermite basis $\{H_k(x)\}_{k=0}^{\infty}$, we will need to generalize a result of Bleecker and Csordas mentioned in the previous section (Proposition 52). The generalization will require the following lemma, which is itself a generalization of a lemma used by Craven and Csordas [10, Lemma 3.8].

Lemma 67. *Suppose that $p(x)$ is a real polynomial of degree n. If c, d, β are real numbers such that $c \geq 0$ and $\beta \geq 0$, then $Z_C\big(q(x)\big) \leq Z_C\big(p(x)\big)$, where*

$$q(x) = (cx + d)p(x) - \beta p'(x). \tag{3.29}$$

Proof. The lemma is clearly true when $\beta = 0$. If $c = 0$, then the lemma is an immediate consequence of the Hermite-Poulain Theorem (Theorem 37).

Suppose $\beta > 0$ and $c > 0$ and let m be the number of real zeros of $p(x)$ counting multiplicities, i.e.,

$$Z_C\big(p(x)\big) = n - m \qquad \big(n = \deg(p)\big). \tag{3.30}$$

We may write the polynomial $q(x)$ of (3.29) in the form

$$q(x) = -\beta \exp\left(\frac{c}{2\beta}x^2 + \frac{d}{\beta}x\right) \frac{d}{dx}\left[\exp\left(-\frac{c}{2\beta}x^2 - \frac{d}{\beta}x\right) p(x)\right].$$

Thus, the zeros of $q(x)$ coincide with the zeros of $f'(x)$, where we define $f(x)$ by

$$f(x) = \exp\left(-\frac{c}{2\beta}x^2 - \frac{d}{\beta}x\right) p(x). \tag{3.31}$$

Furthermore, the zeros of $f(x)$ coincide with the zeros of $p(x)$. Let x_1 and x_2 denote the smallest and largest of the zeros of $f(x)$, respectively. Then, as a consequence of Rolle's theorem (see Corollary 7), $f'(x)$ has at least $m - 1$ real zeros in the interval $[x_1, x_2]$. Furthermore, $f(x)$ vanishes at x_1 and x_2, has constant sign on each of the unbounded intervals $(-\infty, x_1)$ and (x_2, ∞), and tends to zero as $x \to \pm\infty$. Therefore, $f'(x)$ must have at least one zero in each of these unbounded intervals. Whence, $f'(x)$ has at least $(m-1)+2 = m+1$ real zeros. Noting that $\deg(q) = \deg(p)+1$, we have

$$Z_C\big(q(x)\big) = Z_C\big(f'(x)\big) \leq (n+1) - (m+1) = n - m = Z_C\big(p(x)\big).$$

$\square$

We are now in a position to generalize Proposition 52.

Proposition 68. *Suppose*

$$p(x) = \sum_{k=0}^{n} a_k x^k \qquad (a_n \neq 0)$$

is a real polynomial. If α, β, c, d *are real numbers such that* $\alpha \geq 0$*,* $\beta \geq 0$*, and* $\alpha + cn \geq 0$*, then* $Z_C\big(f(x)\big) \leq Z_C\big(p(x)\big)$*, where*

$$f(x) = \alpha p(x) + (cx + d)p'(x) - \beta p''(x).$$

Proof. If $\alpha = 0$, then the condition $\alpha + nc \geq 0$ implies that $c \geq 0$. Thus, by Lemma 67 and Rolle's theorem,

$$Z_C\big(f(x)\big) = Z_C\big((cx + d)p'(x) - \beta p''(x)\big) \leq Z_C\big(p'(x)\big) \leq Z_C\big(p(x)\big).$$

Now suppose $\alpha > 0$. We will first consider the case where each of the real zeros of the polynomial $p(x)$ are simple. Suppose $p(x)$ has exactly m real (simple) zeros, i.e.,

$$Z_C\big(p(x)\big) = n - m \qquad \big(n = \deg(p)\big).$$

Since $\deg(f) \leq \deg(p) = n$, we have $Z_C(f) \leq n$. Thus the proposition is true when $m = 0$. If $m = 1$, then, either $\deg(f) < \deg(p)$, or $f(x)$ is of odd degree and, therefore, has at least one real zero. In either case, $Z_C\big(f(x)\big) \leq n - 1 = Z_C\big(p(x)\big)$, and so the proposition is also true in this case. We may, therefore, assume that $m \geq 2$.

Denote the real zeros of $p(x)$ by $x_1 < x_2 < x_3 < \cdots < x_m$. By Rolle's theorem, for each $i = 1, 2, 3, \ldots, m-1$, $p'(x)$ has at least one real zero in each of the intervals (x_i, x_{i+1}). For each $i = 1, 2, 3, \ldots, m-1$, let y_i denote the *smallest* zero of $p'(x)$ in the interval (x_i, x_{i+1}). To help clarify this notation, we remark that

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{m-1} < y_{m-1} < x_m.$$

In particular, we have $p(y_i) \neq 0$ for $i = 1, 2, 3, \ldots, m-1$. Furthermore, the signs of the sequence $\{p(y_1), p(y_2), p(y_3), \ldots, p(y_{m-1})\}$ alternate.

Fix an integer $i \in \{1, 2, 3, \ldots, m-1\}$. We will demonstrate that $f(y_i) \neq 0$ and that the sign of $f(y_i)$ is the same as the sign of $p(y_i)$. First note that, since x_i is a *simple* zero of $p(x)$, we have $p'(x_i) \neq 0$. Since we have chosen y_i to be the *smallest* zero of $p'(x)$ in the interval (x_i, x_{i+1}), we know that $p'(x)$ has constant sign on $[x_i, y_i)$. Furthermore, since

$$p'(x_i) = \lim_{x \to x_i^+} \frac{p(x) - p(x_i)}{x - x_i} = \lim_{x \to x_i^+} \frac{p(x)}{x - x_i},$$

we have that $p(x)$ and $p'(x)$ have the same sign on the interval (x_i, y_i). It follows that

$$p(y_i)p''(y_i) = p(y_i) \lim_{x \to y_i^-} \frac{p'(x) - p'(y_i)}{x - y_i} = \lim_{x \to y_i^-} \frac{p(y_i)p'(x)}{x - y_i} \leq 0. \qquad (3.32)$$

Therefore, since we have assumed $\alpha > 0$ and $\beta \geq 0$, and since we have chosen y_i so that $p(y_i) \neq 0$ and $p'(y_i) = 0$,

$$f(y_i) = \alpha p(y_i) + (cy_i + d)p'(y_i) - \beta p''(y_i) = \alpha p(y_i) - \beta p''(y_i) \neq 0$$

and

$$\operatorname{sign}\big[f(y_i)\big] = \operatorname{sign}\big[p(y_i)\big] \qquad (i = 1, 2, 3, \ldots m-1). \qquad (3.33)$$

Thus the signs of the sequence

$$\big\{f(y_1), f(y_2), f(y_3), \ldots, f(y_{m-1})\big\}$$

alternate, and, therefore, $f(x)$ has at least $m-2$ zeros in the interval (y_1, y_{m-1}).

Note that the coefficient of x^n in the polynomial $f(x)$ is equal to $(\alpha+cn)a_n$, where a_n is the leading coefficient of $p(x)$. If $\alpha+cn=0$, then $\deg(f)\le n-1$. Thus, in this case,

$$Z_C\big(f(x)\big) \le (n-1)-(m-2) = n-m+1.$$

But, since non-real zeros of a real polynomial come in pairs, we know that $Z_C\big(p\big) = m-n$ is even. Similarly, $Z_C\big(f(x)\big)$ is also even and, therefore, cannot equal $n-m+1$. I.e., in the case where $\alpha+cn=0$, we have shown that $Z_C\big(f(x)\big) \le m-n$.

Now suppose $\alpha+cn>0$. We will demonstrate that $f(x)$ has a zero in the unbounded interval (y_{m-1},∞). Without loss of generality, suppose that the leading coefficient a_n of $p(x)$ is positive. Then $p(y_{m-1})<0$ and, by relation (3.33), $f(y_{m-1})<0$. Since the leading coefficient $(\alpha+cn)a_n$ is also positive, $f(x)\to\infty$ as $x\to\infty$. Therefore, $f(x)$ has a zero in the interval (y_{i+1},∞). Whence

$$Z_C\big(f(x)\big) \le n-(m-1) = n+m+1,$$

and, taking into account the parity of the number of non-real zeros of a real polynomial, we have $Z_C\big(f(x)\big) \le m-n = Z_C\big(p(x)\big)$. It should be noted that one could prove the existence of an additional zero in the other unbounded interval $(-\infty,y_1)$. We have now proved that the proposition is valid when all of the real zeros of $p(x)$ are simple.

In general, suppose $p(x)$ has m real zeros $x_1\le x_2\le x_3\le\cdots\le x_m$, which are not necessarily distinct, and $2N$ non-real zeros $\{\nu_k \pm i\mu_k\}_{k=1}^{N}$. Then, for $\epsilon>0$, we form the polynomial

$$p_\epsilon(x) = a_n \left[\prod_{k=1}^{m} \big(x - (x_k + k\epsilon)\big) \right] \left[\prod_{k=1}^{N} \big((x - \nu_k)^2 + \mu_k^2\big) \right]$$

and set

$$f_\epsilon(x) = \alpha p_\epsilon(x) + (cx + d)p'_\epsilon(x) - \beta p''_\epsilon(x).$$

For each $\epsilon > 0$, all of the real zeros of p_ϵ are simple. Thus

$$Z_C\big(f_\epsilon(x)\big) \le Z_C\big(p_\epsilon(x)\big) = 2N \qquad (\epsilon > 0).$$

The sequence of functions $\left\{f_{1/k}(x)\right\}_{k=1}^{\infty}$ converge uniformly on compact subsets of $\mathbb{C}$ to $f(x)$ and, by Hurwitz' theorem,

$$Z_C\big(f(x)\big) \le 2N = Z_C\big(p(x)\big).$$

$\square$

In particular, Proposition 68 states that the linear operator

$$\alpha + xD - \beta D^2 \qquad (\alpha \ge 0, \beta \ge 0)$$

is a complex zero decreasing operator. We shall use this fact to further generalize Turán's theorem (Theorem 49), but we will postpone this discussion until the next chapter.

Before moving on, it is also worth mentioning another class of complex zero decreasing operators which were studied by Carnicer, Peña, and Pinkus [7] in 2002. They studied linear operators defined by

$$T[x^n] = x^n + \sum_{k=0}^{n-1} b_{n,k} x^n \qquad (b_{n,k} \in \mathbb{R};\ n = 0, 1, 2, \dots) \tag{3.34}$$

and completely characterized all complex zero decreasing operators of this form.

Theorem 69. (Carnicer, Peña, Pinkus [7, p. 5]) *Let the linear operator T be defined by* (3.34). *Then T is a complex zero decreasing operator if and only if $T = F(D)$, where $F(x) \in \mathcal{L} - \mathcal{P}$ and $F(0) = 1$.*

Thus, Carnicer, Peña, and Pinkus characterized complex zero decreasing operators whose corresponding matrix representation (see Remark 35) is unitary upper triangular, i.e., of the form

$$M_T = \begin{pmatrix} 1 & b_{1,0} & b_{2,0} & b_{3,0} & b_{4,0} & \dots \\ 0 & 1 & b_{2,1} & b_{3,1} & b_{4,1} & \dots \\ 0 & 0 & 1 & b_{3,2} & b_{4,2} & \dots \\ 0 & 0 & 0 & 1 & b_{4,3} & \\ 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

However, this class of operators is not applicable to our current investigation since it does not include any of the operators involved in Problem 66 except, of course, the identity operator.

3.5 Extension to Transcendental Entire Functions

Up to now, we have mostly considered linear operators which act on the vector space of real polynomials. In this section, we will consider linear operators which act on transcendental entire functions and, whenever possible, extend the results of the previous sections. In this setting, we must use caution as the issue of convergence often arises. For example, the infinite order differential operator $\sum_{k=0}^{\infty} D^k$ presents no problem when applied to polynomials. However, if $r > 1$ then

$$\left(\sum_{k=0}^{\infty} D^k\right)[e^{rx}] = e^{rx}\sum_{k=0}^{\infty} r^k, \tag{3.35}$$

and the right hand side of (3.35) does not converge for any x. Of course, if we consider a *finite order* differential operator whose coefficients are entire functions, convergence is not an issue. The next lemma demonstrates that operators of this type essentially preserve uniform convergence.

Lemma 70. *Let T be a linear operator on the set of (complex) entire functions and suppose T has the form*

$$T = \sum_{k=0}^{m} T_k(z)D^k,$$

where each function $T_k(z)$ is an entire function and D denotes differentiation with respect to z. If the sequence of entire functions $\{f_n(z)\}_{n=1}^{\infty}$ converge uniformly on compact subsets of $\mathbb{C}$ to the function $f(z)$, then the sequence of entire functions $\{T[f_n(z)]\}_{n=1}^{\infty}$ converge uniformly on compact subsets of $\mathbb{C}$ to the function $T[f(z)]$.

Proof. Let K be a compact subset of $\mathbb{C}$ and suppose $\epsilon > 0$. Since each of the functions $T_k(z)$ are continuous, there exists a constant M such that $|T_k(z)| \leq M$ for all $k = 0, 1, 2, \ldots, m$ and all $z \in K$. Since $f_n \to f$ uniformly on compact subsets of $\mathbb{C}$, we also have that $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of $\mathbb{C}$ (see Theorem 10). Thus, we can pick an integer N such that, for all $k = 0, 1, 2, \ldots, m$,

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| < \frac{\epsilon}{(m+1)M}$$

whenever $n \geq N$ and $z \in K$. Thus, for any $z \in K$ and any $n \geq N$, we have

$$
\begin{aligned}
\left|T[f_n(z)] - T[f(z)]\right| &= \left|\sum_{k=0}^{m} T_k(z) f_n^{(k)}(z) - \sum_{k=0}^{m} T_k(z) f^{(k)}(z)\right| \\
&= \left|\sum_{k=0}^{m} T_k(z)\left(f_n^{(k)}(z) - f^{(k)}(z)\right)\right| \\
&\leq \sum_{k=0}^{m} |T_k(z)| \left|f_n^{(k)}(z) - f^{(k)}(z)\right| \\
&\leq \sum_{k=0}^{m} M \cdot \frac{\epsilon}{(m+1)M} = \epsilon.
\end{aligned}
$$

□

As the next proposition demonstrates, if a finite order differential operator preserves reality of zeros, then this operator maps the entire Laguerre-Pólya class into itself. Since we have defined the concept of preserving reality of zeros for linear operators on the vector space of real polynomials, we need to make the restriction that the coefficients of the differential operator be real polynomials.

Proposition 71. *Let T be a linear operator on the set of (complex) entire functions and suppose T has the form*

$$T = \sum_{k=0}^{m} T_k(z) D^k,$$

where each $T_k(z)$ is a real polynomial and D denotes differentiation with respect to z. If T preserves reality of zeros, then T also has the property that

$$T[\varphi(z)] \in \mathcal{L}-\mathcal{P} \qquad \textit{whenever} \qquad \varphi(z) \in \mathcal{L}-\mathcal{P}.$$

Proof. Suppose the real entire function $\varphi(z)$ belongs to the Laguerre-Pólya class. Then there is a sequence of real polynomials $\{p_n(z)\}_{n=1}^{\infty}$, each of which has only real zeros, which converge uniformly on compact subsets of $\mathbb{C}$ to $\varphi(z)$. By Lemma 70, the sequence of (real) polynomials $\{T[p_n(z)]\}_{n=1}^{\infty}$ converge uniformly on compact subsets of $\mathbb{C}$ to $T[\varphi(z)]$. Furthermore, since we have assumed T preserves reality of zeros, each polynomial $T[p_n(z)]$ has only real zeros (or, perhaps, is identically zero). Therefore, $T[\varphi(z)]$ is the uniform limit on compact subsets of $\mathbb{C}$ of polynomials having only real zeros and, therefore, belongs to the Laguerre-Pólya class (see Remark 18). $\square$

Remark 72. The requirement that $\varphi(x) \in \mathcal{L}-\mathcal{P}$ in Proposition 71 is necessary. Indeed, as was noted in [10, p. 806], the Hermite-Poulain Theorem does not carry over to arbitrary entire functions. For example, $f(x) = \exp(x^2)$ has only real zeros (since it does not have *any* zeros), but $f''(x) = (4x^2+2)\exp(x^2)$ has two non-real zeros.

As the following example demonstrates, Proposition 71 can be used to determine necessary conditions for a differential operator to preserve reality of zeros.

Example 73. Suppose the linear operator T on the set of entire functions is defined by $T = \sum_{k=0}^{m} T_k(x)D^k$, where each $T_k(x)$ is a real polynomial and D denotes differentiation with respect to x. If T preserves reality of zeros, then, for each fixed $r \in \mathbb{R}$,

$$T\left[e^{rx}\right] = e^{rx}\sum_{k=0}^{m} T_k(x)r^k \in \mathcal{L}-\mathcal{P}.$$

Thus, in order that T preserve reality of zeros, it is necessary that, for each fixed $r \in \mathbb{R}$, the polynomial $p_r(x) = \sum_{k=0}^{m} T_k(x)r^k$ belongs to the Laguerre-Pólya class.

Similarly, Proposition 71 provides a way for us to generate new linear operators which preserve reality of zeros from ones we already know.

Example 74. By Laguerre's theorem (Theorem 60), the operator $(1+xD)$ preserves reality of zeros. Thus, if $p(x)$ is a polynomial having only real zeros, then

$$(1+xD)\,[p(x)e^x] = e^x\big((x+1)p(x) + xp'(x)\big) \in \mathcal{L}-\mathcal{P}.$$

Thus $(x+1)p(x) + xp'(x)$ has only real zeros, i.e., the linear operator $(x+1)+xD$ preserves reality of zeros.

Finally, we remark that *infinite order* differential operators with *constant* coefficients have been studied by several authors. For several results on operators of this form which act on transcendental entire functions, we refer the reader to [20, Chapter IX], [10], and the references therein.

Chapter 4
Zeros of Hermite Expansions

4.1 Heuristic Principles for Zeros in a Strip

The Hermite polynomials have received a lot of attention from us thus far, and one may wonder why we have chosen to study these polynomials. In 1950, Turán [29] noted that, in studying the distribution of zeros of polynomials in a strip in the complex plane, it may be advantageous to examine the Hermite expansion of a polynomial. The heuristic principle that Turán used is as follows. If we write a polynomial in the form $p(x) = \sum_{k=0}^{n} a_k x^k$, then, from the coefficients a_k, one can determine certain circular regions which contain all the zeros of $p(x)$. Turán believed that this was a consequence of the fact that the curves $|z^n| = c$, where $c \geq 0$ are concentric circles. Thus, if one wanted to determine whether or not all the zeros of a given polynomial lie in a certain strip in the complex plane which is symmetric about the real axis, then one should write the polynomial in terms of polynomials $\{p_k(z)\}_{k=0}^{\infty}$ which have the property that the curves $|p_k(z)| = c$, where $c \geq 0$ are *essentially* horizontal lines. Since, for large n, the n^{th} Hermite polynomial fits this description, the use of these polynomials in the study of the distribution of zeros of a function in a strip (or, in particular, on the real axis) seemed natural. Consider, for example, the classical results regarding zeros of polynomials in a circle.

Theorem 75. *Let* $p(z)=\sum_{k=0}^{n}a_kz^k$ *be a complex polynomial, where* $a_n\neq 0$.

(i) (Cauchy [22, p. 96]) *All of the zeros of* $p(z)$ *lie in the circle*

$$|z|\leq 1+A_1 \qquad \left(A_1=\max\left\{\left|\frac{a_k}{a_n}\right| : 0\leq k\leq n-1\right\}\right).$$

(ii) (Walsh [22, p. 98]) *All of the zeros of* $p(z)$ *lie in the circle*

$$|z|\leq A_2 \qquad \left(A_2=\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^{\frac{1}{n-k}}\right).$$

Turán discovered the following analogues for zeros in a strip.

Theorem 76. *Let* $q(z)=\sum_{k=0}^{n}b_kH_k(z)$ *be a complex polynomial, where* H_k *denotes the* k^{th} *Hermite polynomial and* $b_n\neq 0$.

(i) (Turán [29, p. 283] and [30]) *The zeros of* $q(z)$ *lie in the strip*

$$|\mathrm{Im}\ z|\leq \frac{1}{2}(1+B_1) \qquad \left(B_1=\max\left\{\left|\frac{b_k}{b_n}\right| : 0\leq k\leq n-1\right\}\right).$$

(ii) (Turán [29, p. 284] and [30]) *The zeros of* $q(z)$ *lie in the strip*

$$|\mathrm{Im}\ z|\leq B_2 \qquad \left(B_2=\sum_{k=0}^{n-1}\left|\frac{b_k}{b_n}\right|^{\frac{1}{n-k}}\right).$$

Turán's proof of Theorem 76 was based on an certain inequality which is satisfied by the Hermite polynomials. However, we will employ more sophisticated methods to obtain a better result. Before reviewing these methods, let us introduce a notation which will facilitate the discussion.

Notation 77. For $A \geq 0$, we will denote by $S(A)$ the strip in the complex plane of width $2A$ which is symmetric about the real axis.

$$S(A) = \{z \in \mathbb{C} \,:\, |\mathrm{Im}\ z| \leq A\} \qquad (A \geq 0).$$

In 2001, Bleecker and Csordas [1] studied the distribution of zeros of Hermite expansions in a strip. Their investigation was guided by the another heuristic principle which can be expressed as follows. If $t > 0$, then, under the action of the linear operator $\exp(-tD^2)$, the zeros of a polynomial tend to be attracted to the real axis, while, under the action of the linear operator $\exp(tD^2)$, the zeros of a polynomial tend to be repelled by the real axis (see [1, p. 190]). To demonstrate a theorem in support of this heuristic principle, they employed a result of DeBruijn [4, Theorem 5] to obtain the following result.

Theorem 78. (Bleecker-Csordas [1, p. 191]) *If all the zeros of the non-constant polynomial* $p(x) = \displaystyle\sum_{k=0}^{n} a_k x^k$ *lie in the strip* $S(A)$*, for some* $A \geq 0$*, then, for each fixed* $t > 0$*, the zeros of the polynomial*

$$e^{-tD^2}\left[p(x)\right] = \sum_{k=0}^{n} a_k t^{k/2} H_k\left(\frac{x}{2\sqrt{t}}\right) \tag{4.1}$$

lie in the strip $S\left(\sqrt{\max\{A^2 - 2t, 0\}}\right)$.

It should be noted that the equality in equation (4.1) is an immediate consequence of equation (3.15) of Example 30.

A particular consequence of Theorem 78 is the following corollary which shows that the zeros of a polynomial are attracted to the real axis under the action of the linear operator defined by $T[x^n] = H_n(x)$.

Corollary 79. *If, for some $A \geq 0$, the zeros of $p(x) = \sum_{k=0}^{n} a_k x^k$ lie in the strip $S(A)$, then the zeros of $q(x) = \sum_{k=0}^{n} a_k H_k(x)$ lie in the strip $S\left(\frac{1}{2}\sqrt{\max\{A^2 - 2, 0\}}\right)$. In particular, if $A \leq \sqrt{2}$, then all of the zeros of $q(x)$ are real.*

Proof. Taking $t = 1$ in Theorem 78, we have that the zeros of

$$f(x) = e^{-D^2} p(x) = \sum_{k=0}^{n} a_k H_k\left(\frac{x}{2}\right)$$

lie in the strip $S\left(\sqrt{\max\{A^2 - 2, 0\}}\right)$. Thus, the zeros of the polynomial

$$f(2x) = \sum_{k=0}^{n} a_k H_k(x) = q(x)$$

lie in the strip $S\left(\frac{1}{2}\sqrt{\max\{A^2 - 2, 0\}}\right)$. □

Remark 80. This result is the best possible in the following sense. For any $A \geq 0$, the zeros of $x^2 + A^2$ lie in the strip $S(A)$ and the zeros of

$$H_2(x) + A^2 H_0(x) = 4x^2 - 2 + A^2$$

lie on the boundary of the strip $S\left(\frac{1}{2}\sqrt{\max\{A^2 - 2, 0\}}\right)$.

Combining the classical results of Cauchy and Walsh stated in Theorem 75 with Corollary 79, we are able to refine Theorem 76 as follows.

Corollary 81. *Let $q(z) = \sum_{k=0}^{n} b_k H_k(z)$ be a complex polynomial, where H_k denotes the k^{th} Hermite polynomial and $b_n \neq 0$.*

(i) *The zeros of $q(z)$ lie in the strip $S(C_1)$, where*

$$C_1 = \frac{1}{2}\sqrt{\max\left\{(1 + B_1)^2 - 2, 0\right\}} \qquad \left(B_1 = \max\left\{\left|\frac{b_k}{b_n}\right| : 0 \leq k \leq n-1\right\}\right).$$

(ii) *The zeros of $q(x)$ lie in the strip $S(C_2)$, where*

$$C_2 = \frac{1}{2}\sqrt{\max\{(B_2)^2 - 2,\, 0\}} \qquad \left(B_2 = \sum_{k=0}^{n-1} \left|\frac{b_k}{b_n}\right|^{\frac{1}{n-k}}\right).$$

In particular, if $B_1 \leq (\sqrt{2}-1)$ or $B_2 \leq \sqrt{2}$, then $q(z)$ has only real zeros.

4.2 Sufficient Conditions for Reality of Zeros

Corollaries 79 and 81 provide, in particular, sufficient conditions for a polynomial to have only real zeros. Let us now combine these results with a well-known classical result to obtain another sufficient condition for the reality of zeros of a polynomial in terms of the coefficients its Hermite expansion.

Theorem 82. (Eneström-Kakeya Theorem [22, p. 106]) *If*

$$0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,$$

then all of the zeros of the polynomial $\displaystyle\sum_{k=0}^{n} a_k z^k$ lie in the circle $|z| \leq 1$.

Corollary 83. *If*

$$0 < b_0 \leq \sqrt{2} b_1 \leq (\sqrt{2})^2 b_2 \leq \cdots \leq (\sqrt{2})^n b_n,$$

then the polynomial $p(x) = \displaystyle\sum_{k=0}^{n} b_k H_k(x)$ has only real zeros.

Proof. By the Eneström-Kakeya Theorem, all of the zeros of

$$q(x) = \sum_{k=0}^{n} (\sqrt{2})^k b_k x^k$$

lie in the unit circle. Thus all of the zeros of

$$q\left(\frac{x}{\sqrt{2}}\right) = \sum_{k=0}^{n} b_k x^k$$

lie in the circle $\left\{z : |z| \leq \sqrt{2}\right\}$ which is contained in the strip $S(\sqrt{2})$. Therefore, by Corollary 79, the zeros of the polynomial

$$p(x) = \sum_{k=0}^{n} b_k H_k(x)$$

are all real. □

Although we will not include them here, similar results can be obtained by combining any of the classical results concerning zeros in a circle with Corollary 79. For example, there are several known generalizations of the Eneström-Kakeya Theorem (Theorem 82) which can be used to obtain similar results. For examples of these generalizations, see [13] and the references contained therein.

In his 1950 paper [29], Turán announced some results and also posed some questions regarding the reality of zeros of polynomials in terms of the coefficients of their Hermite expansions. The first such result is as follows.

Theorem 84. (Turán [29, p. 286], [31, p. 127], and [23, p. 207]) *If the set of real numbers $\{b_k\}_{k=0}^{n}$ satisfy the inequality*

$$\sum_{k=0}^{n-2} 2^k k!\, b_k^2 < 2^n (n-1)!\, b_n^2\,, \tag{4.2}$$

then the zeros of the real polynomial $p(x) = \sum_{k=0}^{n} b_k H_k(x)$ are real, simple, and separated by the zeros of $H_{n-1}(x)$.

It is somewhat remarkable that the coefficient b_{n-1} plays no role in Theorem 84. The reason for this is that the theorem was proved by employing the Christoffel-Darboux formula (2.9) and the Cauchy-Schwarz inequality to examine the sign of the polynomial in question at the zeros of $H_{n-1}(x)$. Since the $b_{n-1}H_{n-1}(x)$ term of $f(x)$ vanishes at these zeros, the coefficient b_{n-1} has no effect on the result.

In connection with the Riemann Hypothesis, Turán was particularly interested in the distribution of zeros of polynomials of the form

$$q(x) = \sum_{k=0}^{n} (-1)^k b_{2k} H_{2k}(x) \qquad (b_{2k} > 0 \text{ for } k = 0, 1, 2, \ldots, n). \tag{4.3}$$

Turán [29, p. 286] declared that, if the coefficients b_{2k} of the polynomial $q(x)$ defined by (4.3) satisfy

$$\frac{b_{2k+2}}{b_{2k}} > \frac{1}{4} \qquad (k = 0, 1, 2, \ldots, n-1), \tag{4.4}$$

then $q(x)$ must have only real zeros. Almost a decade later (see [31, p. 131]), it was claimed that this fact follows from Theorem 84. That is to say, it was claimed that, if the set of positive real numbers $\{b_{2k}\}_{k=0}^{n}$ satisfy (4.4), then

$$\sum_{k=0}^{n-1} 2^{2k}(2k)!\, b_{2k}^2 < 2^{2n}(2n-1)!\, b_{2n}^2\,. \tag{4.5}$$

However, this is not true. For example, if we take $b_0 = 1$, $b_2 = 3/8$, and $b_4 = 1/8$, then (4.4) is satisfied, but (4.5) is not. It can be shown that the claim does hold whenever $n \geq 4$, and one can treat the cases $n = 1$, 2, and 3 with other methods. Perhaps this is what Turán originally intended and, in any case, the claim was immediately followed by a more general theorem which was proved using different techniques (see [31, p. 131]).

Bleecker and Csordas [1, Problem 4.5] asked if the constant $1/4$ in (4.4) was best possible. In what follows, we will show that constant $1/4$ is *not* best possible, and we will also provide a lower bound for the best possible constant. To facilitate the discussion, let us restate the problem as follows.

Problem 85. What is the largest real number $L > 0$ such that, for any natural number $n \geq 1$, the polynomial

$$q(x) = \sum_{k=0}^{n} (-1)^k b_{2k} H_{2k}(x), \tag{4.6}$$

where H_n denotes the n^{th} Hermite polynomial, has only real zeros whenever

$$0 < b_0 \leq L\, b_2 \leq L^2\, b_4 \leq \cdots \leq L^n\, b_{2n}\,? \tag{4.7}$$

Remark 86. There are three items regarding Problem 85 which require attention.

1. Note that the condition (4.7) is equivalent to saying that $b_{2k} > 0$ for all $k = 0, 1, 2, \ldots, n$ and

$$\frac{b_{2k+2}}{b_{2k}} \geq \frac{1}{L} \qquad (k = 0, 1, 2, \ldots, n-1).$$

2. Suppose the real number $L > 0$ has the property that, for every natural number $n \geq 1$, the polynomial $q(x)$ of (4.6) has only real zeros whenever

$$0 < b_0 < L\, b_2 < L^2\, b_4 < \cdots < L^n\, b_{2n}.$$

Then $q(x)$ also has only real zeros whenever

$$0 < b_0 \leq L\, b_2 \leq L^2\, b_4 \leq \cdots \leq L^n\, b_{2n}. \tag{4.8}$$

Indeed, if (4.8) holds, then for any $\epsilon > 0$

$$0 < b_0 < L\left(b_2 + \frac{\epsilon}{L}\right) < L^2\left(b_4 + \frac{2\epsilon}{L^2}\right) < \cdots < L^n\left(b_{2n} + \frac{n\epsilon}{L^n}\right).$$

Thus, for any $\epsilon > 0$, the polynomial

$$q_\epsilon(x) = \sum_{k=0}^{n}(-1)^k\left(b_{2k} + \frac{k\epsilon}{L^k}\right)H_{2k}(x)$$

has only real zeros. Since the sequence of polynomials $\left\{q_{1/N}(x)\right\}_{N=1}^{\infty}$ converge uniformly on compact subsets to $q(x)$, we have, by Hurwitz' theorem, that $q(x)$ has only real zeros.

3. It is interesting to note that Corollary 83 can be used to show that $L \geq 2$. Also, in light of the second item in this remark, Turán's result shows that $L \geq 4$. We will show that we can actually take L to be larger than 4 but that we cannot take L to be larger than a certain number which is approximately equal to 8.3.

The following lemma will be instrumental in providing an upper bound for L of Problem 85. It will also be used to prove that we may take L to be larger than 4. The calculations involved are somewhat tedious but, to quote E. Makai and P. Turán [21, p. 143], "a first proof can be as ugly as it wants to be."

Lemma 87. *Let $\{b_0, b_2, b_4, b_6\}$ be a set of positive real numbers. Then*

(i) $f_1(x) = b_0H_0(x) - b_2H_2(x)$ *has only real zeros.*

(ii) $f_2(x) = b_0H_0(x) - b_2H_2(x) + b_4H_4(x)$ *has only real zeros if and only if*

$$\Delta_2 = b_2^2 + 16b_2b_4 + 96b_4^2 - 4b_0b_4 \geq 0 \tag{4.9}$$

(iii) $f_3(x) = b_0 H_0(x) - b_2 H_2(x) + b_4 H_4(x) - b_6 H_6(x)$ *has only real zeros if and only if* $\Delta_3 \leq 0$, *where*

$$\begin{aligned} \Delta_3 \;=\; & 27b_0^2b_6^2 + 4b_0b_4^3 + 144b_0b_4^2b_6 + 1728b_0b_4b_6^2 + 17280b_0b_6^3 + 4b_2^3b_6 + 48b_2^2b_4b_6 \\ & +288b_2^2b_6^2 + 3456b_2b_4b_6^2 + 34560b_2b_6^3 - 96b_4^4 - 3840b_4^3b_6 - 76032b_4^2b_6^2 \\ & -829440b_4b_6^3 - 4147200b_6^4 - 18b_0b_2b_4b_6 - 432b_0b_2b_6^2 - b_2^2b_4^2 - 16b_2b_4^3 \end{aligned} \tag{4.10}$$

Proof. For (i), note that

$$f_1(x) = b_0H_0(x) - b_2H_2(x) = b_0(1) - b_2(4x^2 - 2) = -4b_2x^2 + 2b_2 + b_0.$$

Whence the result follows immediately from our assumption that $b_0, b_2 > 0$.

For (ii), we note that

$$\begin{aligned} f_2(x) &= b_0H_0(x) - b^2H_2(x) + b_4H_4(x) \\ &= b_0(1) - b_2(4x^2 - 2) + b_4(16x^4 - 48x^2 + 12) \\ &= 16b_4x^4 - 4(b_2 + 12b_4)x^2 + b_0 + 2b_2 + 12b_4. \end{aligned}$$

Since the signs of the coefficients of the polynomial

$$f_2\big(\sqrt{x}\big) = 16b_4x^2 - 4(b_2 + 12b_4)x + b_0 + 2b_2 + 12b_4$$

alternate, $f_2\big(\sqrt{x}\big)$ has two positive zeros whenever its discriminant is non-negative. Thus $f_2(x)$ has only real zeros whenever $f_2\big(\sqrt{x}\big)$ does, i.e., whenever

$$16(b_2 + 12b_4)^2 - 4(16b_4)(b_0 + 2b_2 + 12b_4) \geq 0,$$

which is equivalent to the condition $\Delta_2 \geq 0$ of equation (4.9).

In case (iii), a calculation shows that

$$f_3(x) = -64b_6x^6 + 16(b_4 + 30b_6)x^4 - 4(b_2 + 12(b_4 + 15b_6))x^2 + b_0 + 2b_2 + 12b_4 + 120b_6.$$

Since the signs of coefficients of the polynomial

$$f_3\big(\sqrt{x}\big) = -64b_6x^3 + 16(b_4 + 30b_6)x^2 - 4(b_2 + 12(b_4 + 15b_6))x + b_0 + 2b_2 + 12b_4 + 120b_6$$

alternate, we know that if $f_3\big(\sqrt{x}\big)$ has only real zeros then they must all be positive. Therefore, $f_3(x)$ has only real zeros whenever $f_3\big(\sqrt{x}\big)$ does. A cubic polynomial

$$a_0 + a_1x + a_2x^2 + a_3x^3$$

has only real zeros if and only if its discriminant

$$\Delta = \frac{a_3^2(27a_0^2a_3^2 + 2a_0a_2(2a_2^2 - 9a_1a_3) + 4a_1^3a_3 - a_1^2a_2^2)}{27}$$

is non-positive (see, for example, [5, p. 42]). A calculation shows that the discriminant of the cubic polynomial $f_3\big(\sqrt{x}\big)$ is $\Delta = \big(2^{24}/3^3\big)\Delta_3$ and, therefore, $f_3(x)$ has only real zeros if and only if $\Delta_3 \leq 0$. $\square$

We can now give an upper bound for the value of L in Problem 85.

Proposition 88. *Suppose the real number $L > 0$ is such that, for any natural number $n \geq 1$, the polynomial*

$$q(x) = \sum_{k=0}^{n}(-1)^k b_{2k}H_{2k}(x), \tag{4.11}$$

where H_n denotes the n^{th} Hermite polynomial, has only real zeros whenever

$$0 < b_0 \leq L\, b_2 \leq L^2\, b_4 \leq \cdots \leq L^n\, b_{2n}.$$

Then $L \leq L_0$, where $L_0 \approx 8.29226323$ is the smallest positive zero of the polynomial

$$x^6 - 16x^5 + 120x^4 + 1056x^3 - 2592x^2 - 51840x - 259200. \tag{4.12}$$

Proof. By Descartes rule of signs, the polynomial (4.12) either has 3 positive zeros, or less by an even number. Thus, the assertion that (4.12) has a (smallest) positive zero is valid.

By hypothesis, the zeros of the polynomial

$$f(x) = H_0(x) - \frac{1}{L}H_2(x) + \frac{1}{L^2}H_4(x) - \frac{1}{L^3}H_6(x)$$

are all real. Thus, as a calculation shows, by taking $b_{2k} = 1/L^k$ for $k = 0$, 1, 2, 3 in equation (4.10), we have that

$$\Delta_3 = \frac{16}{L^{12}}(L^6 - 16L^5 + 120L^4 + 1056L^3 - 2592L^2 - 51840L - 259200) \leq 0. \quad (4.13)$$

It can be shown that (4.12) actually has exactly one real zero. Since, as a polynomial in L,

$$L^6 - 16L^5 + 120L^4 + 1056L^3 - 2592L^2 - 51840L - 259200 \qquad \big(L \in [0, \infty)\big)$$

is negative when $L = 0$, it must be positive on the interval $[L_0, \infty)$. Therefore, by inequality (4.13), we must have $L \leq L_0$. $\square$

Let us now provide a lower bound for L, which is larger than 4. We begin by investigating several special cases.

Lemma 89. *If* $0 < L \leq 2 + 2\sqrt{7}$ *and* $0 < b_0 \leq L\, b_2 \leq L^2\, b_4$*, then the polynomial*

$$f_2(x) = b_0 H_0(x) - b^2 H_2(x) + b_4 H_4(x)$$

has only real zeros. We note that $2 + 2\sqrt{7}$ *is approximately equal to* 7.291502622.

Proof. By Lemma 87, it suffices to show

$$\Delta_2 = b_2^2 + 16b_2b_4 + 96b_4^2 - 4b_0b_4 \geq 0.$$

Since b_0, b_2, and b_4 are positive and $b_0 \leq L\,b_2$,

$$\begin{aligned} \Delta_2 &= b_2^2 + 16b_2b_4 + 96b_4^2 - 4b_0b_4 \\ &\geq b_2^2 + 16b_2b_4 + 96b_4^2 - 4(Lb_2)b_4 \\ &= b_2^2 + (16 - 4L)b_2b_4 + 96b_4^2. \end{aligned}$$

If $L \leq 4$, then Δ_2 is clearly positive. If $L > 4$, then

$$\begin{aligned} \Delta_2 &\geq b_2^2 + (16 - 4L)b_2b_4 + 96b_4^2 \\ &\geq b_2^2 + (16 - 4L)(Lb_4)b_4 + 96b_4^2 \\ &= b_2^2 - 4(L^2 - 4L - 24)b_4^2. \end{aligned}$$

Since we have assumed $0 < L \leq 2 + 2\sqrt{7}$, we have

$$L^2 - 4L - 24 = \big(L - (2 - 2\sqrt{7})\big)\big(L - (2 + 2\sqrt{7})\big) \leq 0$$

and, therefore, $\Delta_2 \geq 0$. $\square$

Remark 90. Actually, since $b_2 > 0$, we have $\Delta_2 > 0$ and so the upper bound $2+2\sqrt{7}$ for L in Lemma 89 is not the best possible. However, this bound will suffice for our current goal.

Lemma 91. *Let L_2 denote the only real zero of the cubic polynomial*

$$x(x(x + 2\sqrt{10} - 8) + 32 - 16\sqrt{10}) + 320 - 160\sqrt{10}. \tag{4.14}$$

If $0 < L < L_2$ and $0 < b_0 \leq L\,b_2 \leq L^2\,b_4 \leq L^3\,b_6$, then the polynomial

$$f_3(x) = b_0H_0(x) - b^2H_2(x) + b_4H_4(x) - b_6H_6(x)$$

has only real zeros. We note that L_2 is approximately equal to 7.48240361.

Proof. We will examine the sign of the polynomial $f_3(x)$ at the zeros of the polynomial $H_5(x)$. Since $H_n(x)$ is an even polynomial for even n, $f_3(x)$ is also an even polynomial. Thus, it suffices to count zeros on the positive real axis. Let $x_0 < x_1 < x_2$ denote the non-negative zeros of $H_5(x)$, i.e.,

$$x_0 = 0 \qquad x_1 = \sqrt{\frac{5-\sqrt{10}}{2}} \qquad x_2 = \sqrt{\frac{5+\sqrt{10}}{2}}.$$

Since the proof will get quite technical, it is important to keep in mind that our goal is to show

$$f(x_0) > 0, \quad f(x_1) < 0, \quad f(x_2) > 0, \quad \text{and} \quad f(\infty) = -\infty. \tag{4.15}$$

A calculation shows that

$$f(x_0) = f(0) = b_0 + 2b_2 + 12b_4 + 120b_6.$$

Since our assumptions imply that b_0, b_2, b_4, and b_6 are positive, we have $f(x_1) > 0$. Another calculation shows that

$$f_3(x_1) = b_0 + (2\sqrt{10} - 8)b_2 + (32 - 16\sqrt{10})b_4 + (320 - 160\sqrt{10})b_6, \tag{4.16}$$

and we note that $(2\sqrt{10} - 8)$, $(32 - 16\sqrt{10})$, and $(320 - 160\sqrt{10})$ are all negative. The assumption $0 < b_0 \leq L\, b_2$, together with equation (4.16), yields the inequality

$$f_3(x_1) \leq (L + 2\sqrt{10} - 8)b_2 + (32 - 16\sqrt{10})b_4 + (320 - 160\sqrt{10})b_6.$$

If $L + 2\sqrt{10} - 8 \leq 0$, which occurs when $L \leq 8 - 2\sqrt{10} \approx 1.675444679$, then we clearly have $f_3(x_1) < 0$. Otherwise, the relation $b_2 \leq L\, b_4$ yields

$$\begin{aligned} f_3(x_1) &\leq (L + 2\sqrt{10} - 8)b_2 + (32 - 16\sqrt{10})b_4 + (320 - 160\sqrt{10})b_6 \\ &\leq \left(L(L + 2\sqrt{10} - 8) + 32 - 16\sqrt{10}\right) b^4 + (320 - 160\sqrt{10})b_6. \end{aligned}$$

Now, if

$$L(L + 2\sqrt{10} - 8) + 32 - 16\sqrt{10} \leq 0,$$

which occurs when $L \leq \sqrt{8\sqrt{10} - 6} - \sqrt{10} + 4 \approx 5.230696421$, then we have $f_3(x_1) < 0$. Otherwise, the relation $b_4 \leq L\, b_6$ yields

$$\begin{aligned} f_3(x_1) &\leq \left(L(L + 2\sqrt{10} - 8) + 32 - 16\sqrt{10}\right) b^4 + (320 - 160\sqrt{10})b_6 \\ &\leq \left(L\left(L(L + 2\sqrt{10} - 8) + 32 - 16\sqrt{10}\right) + 320 - 160\sqrt{10}\right) b_6. \end{aligned}$$

Finally, we note that if

$$L\left(L(L + 2\sqrt{10} - 8) + 32 - 16\sqrt{10}\right) + 320 - 160\sqrt{10} < 0,$$

which occurs whenever $L < L_2$ (recall that L_2 is, by definition, the only positive zero of (4.14)), then $f_3(x_1) < 0$.

We will now investigate the behavior of $f_3(x)$ at x_2. A calculation shows

$$f_3(x_2) = b_0 - (2\sqrt{10} + 8)b_2 + (16\sqrt{10} + 32)b_4 + (160\sqrt{10} + 320)b_6.$$

Thus,

$$f_3(x_2) \geq b_0 + \left(-L(2\sqrt{10} + 8) + (16\sqrt{10} + 32)\right) b_4 + (160\sqrt{10} + 320)b_6.$$

If $-L(2\sqrt{10} + 8) + (16\sqrt{10} + 32) \geq 0$, which occurs whenever

$$L < \frac{8(\sqrt{10} - 1)}{3} \approx 5.766073760,$$

then $f_3(x_2) > 0$. Otherwise, we have

$$\begin{aligned} f_3(x_2) &\geq b_0 + \left(-L(2\sqrt{10} + 8) + (16\sqrt{10} + 32)\right) b_4 + (160\sqrt{10} + 320)b_6 \\ &\geq b_0 + \left(L\left(-L(2\sqrt{10} + 8) + (16\sqrt{10} + 32)\right) + (160\sqrt{10} + 320)\right) b_6. \end{aligned}$$

Now, if

$$L\left(-L(2\sqrt{10}+8)+(16\sqrt{10}+32)\right)+(160\sqrt{10}+320)\geq 0,$$

which occurs whenever

$$L\leq\frac{88(\sqrt{10}-1)}{3}\approx 63.42681136,$$

then $f_3(x_2)>0$. Finally we note that the leading coefficient of $f_3(x)$ is negative. Therefore, $f_3(\infty)=-\infty$.

We have now demonstrated that, for $0<L<L_2$, the four inequalities (4.15) hold and, therefore, $f_3(x)$ has at least 3 positive real zeros. Since $f_3(x)$ is an even polynomial of degree 6, $f_3(x)$ has only real zeros. □

The next lemma covers the remainder of the cases for our current goal.

Lemma 92. *Let L_1 denote the only positive zero of the polynomial*

$$p(x)=x^8+2^2 2!x^6+2^4 4!x^4+2^6 6!x^2-2^8 7!$$

and suppose $0<L\leq L_1\approx 4.462376406$. If the sequence of positive real numbers $\{b_{2k}\}_{k=0}^{\infty}$ satisfy the condition

$$0<b_0<L\,b_2<L^2\,b_4<L^3 b_6<\cdots$$

then, for any integer $n\geq 4$,

$$\sum_{k=0}^{n-1}2^{2k}(2k)!\,b_{2k}^2<2^{2n}(2n-1)!\,b_{2n}^2. \tag{4.17}$$

Proof. We first note that

$$p'(x)=8x^7+2^2 2!6x^5+2^4 4!4x^3+2^6 6!2x$$

has exactly one real zero, which is at the origin. Since $p'(x)$ is a polynomial of odd degree with positive leading coefficient, it follows that $p'(x)$ is negative on the interval $(-\infty, 0)$ and positive on the interval $(0, \infty)$. Since $p(0) < 0$, the assertion that $p(x)$ has exactly one positive zero is valid. Furthermore, $p(x) \leq 0$ on the interval $[0, L_1]$, from which we have the inequality

$$x^8 + 2^2 2! x^6 + 2^4 4! x^4 + 2^6 6! x^2 \leq 2^8 7! \qquad (x \in [0, L_1]).$$

We will prove the proposition by induction on n. First we consider the case when $n = 4$. In this case, we have

$$\begin{aligned}
\sum_{k=0}^{3} 2^{2k} (2k)!\, b_{2k}^2 &= b_0^2 + 2^2 2! b_2^2 + 2^4 4! b_4^2 + 2^6 6! b_6^2 \\
&< L^8 b_8^2 + 2^2 2! L^6 b_8^2 + 2^4 4! L^4 b_8^2 + 2^6 6! L^2 b_8^2 \\
&= \left(L^8 + 2^2 2! L^6 + 2^4 4! L^4 + 2^6 6! L^2\right) b_8^2 \\
&\leq 2^8 7! b_8^2,
\end{aligned}$$

which is what was to be shown. Now suppose inequality (4.17) holds for some given integer $n \geq 4$. Then

$$\begin{aligned}
\sum_{k=0}^{n} 2^{2k} (2k)!\, b_{2k}^2 &= \sum_{k=0}^{n-1} 2^{2k} (2k)!\, b_{2k}^2 + 2^{2n} (2n)! b_{2n}^2 \\
&< 2^{2n} (2n-1)!\, b_{2n}^2 + 2^{2n} (2n)! b_{2n}^2 \\
&= 2^{2n} (2n-1)!\, (1+2n)\, b_{2n}^2.
\end{aligned} \tag{4.18}$$

Since we have assumed $b_{2n} < L b_{2n+2}$, the inequality (4.18) becomes

$$\begin{aligned}
\sum_{k=0}^{n} 2^{2k} (2k)!\, b_{2k}^2 &< 2^{2n} (2n-1)!\, (2n+1)\, L^2 b_{2n+2}^2 \\
&= 2^{2n+2} (2n+1)! b_{2n+2}^2 \frac{L^2}{8n}
\end{aligned} \tag{4.19}$$

Noting that $8n \geq 32$, and $L < 5$, inequality (4.19) becomes

$$\sum_{k=0}^{n} 2^{2k}(2k)!\, b_{2k}^2 < 2^{2n+2}(2n+1)! b_{2n+2}^2,$$

as desired. □

We can now give a slightly better lower bound for the value of L in Problem 85.

Proposition 93. *Let L_1 denote the only positive zero of the polynomial*

$$p(x) = x^8 + 2^2 2! x^6 + 2^4 4! x^4 + 2^6 6! x^2 - 2^8 7!$$

($L_1 \approx 4.462376406$) *and suppose the sequence of positive real numbers $\{b_{2k}\}_{k=0}^{\infty}$ satisfy the condition*

$$0 < b_0 \leq L_1\, b_2 \leq L_1^2\, b_4 \leq L_1^3 b_6 < \cdots .$$

Then, for every non-negative integer n, the polynomial

$$\sum_{k=0}^{n} (-1)^k b_{2k} H_{2k}(x)$$

has only real zeros.

Proof. By Lemma 87 the proposition holds for $n = 0$ and $n = 1$. By Lemmas 89 and 91, the proposition holds for $n = 2$ and $n = 3$. Finally, for $n \geq 4$, the proposition follows from Lemma 92 and Turán's theorem regarding a sufficient condition for the reality of zeros of a polynomial in terms of its Hermite expansion coefficients (Theorem 84). We note that, although the inequality in Theorem 84 is strict, a limiting argument shows that it is also valid when the inequality is non-strict. □

Remark 94. Suppose $0 < b_0 \leq Lb_2 \leq L^2 b_4 \leq L^3 b_6 \leq \cdots$. Then, as in Lemma 91, by examining the sign of the function

$$f_4(x) = b_0 H_0(x) - b_2 H_2(x) + b_4 H_4(x) - b_6 H_6(x) + b_8 H_8(x)$$

at the non-negative zeros of the polynomial $H_7(x)$, one could determine that a sufficient condition for $f_4(x)$ to have only real zeros is that $L \leq 8.4$. Then one could argue, as in Lemma 92, that the inequality

$$\sum_{k=0}^{n-1} 2^{2k}(2k)!\, b_{2k}^2 < 2^{2n}(2n-1)!\, b_{2n}^2$$

holds whenever $n \geq 5$, provided L is less than or equal to the only positive zero of the polynomial

$$x^{10} + 2^2 2! x^8 + 2^4 4! x^6 + 2^6 6! x^4 + 2^8 8! x^2 - 2^{10} 9!,$$

which is approximately equal to 5.374797852. In this way, one could prove that L in Problem 85 could be taken to be as big as this positive zero. However, we will not provide the details of this as they are quite tedious. Instead, we note that, just as in the Turán's proof of Theorem 84, we are examining the polynomial $f_n(x)$ at the zeros of $H_{2n-1}(x)$. This leads us to the following problem.

Problem 95. Can the inequality (4.2) in Theorem 84 be relaxed in the case where the polynomial $p(x)$ is of the form

$$p(x) = \sum_{k=0}^{n} (-1)^k b_{2k} H_{2k}(x)?$$

Next, we would like to answer a problem that was raised by Turán in [29] and again over a decade later in [32]. Although the solution is not that remarkable in itself, we mention it here due to the fact that, until now, it does not seem to have been addressed in the literature.

Problem 96. (Turán [29, p. 297] and [32, p. 419]) Is it true that the polynomial

$$p(x) = \sum_{k=0}^{n} \frac{(-1)^k a_k}{2^{2k}\,(2k)!} H_{2k}(x),$$

where $H_n(x)$ is the n^{th} Hermite polynomial, has only real zeros whenever

$$0 < a_0 < a_1 < a_2 < \cdots < a_n?$$

The answer to this problem is no. Consider, for example, the polynomial

$$p(x) = H_0(x) - \frac{2}{2^2 2!} H_2(x) + \frac{3}{2^4 4!} H_4(x) - \frac{6}{2^6 6!} H_6(x).$$

A calculation shows that the derivative of $p(x)$ is

$$p'(x) = -\frac{1}{80} x\left(4x^4 - 60x^2 + 235\right)$$

which, since $60^2 - 4(4)(235) < 0$, does *not* have only real zeros. Therefore $p(x)$ cannot have all its zeros real.

4.3 Hermite Complex Zero Decreasing Sequences

In this section we will investigate the problem of characterizing all real sequences which are H-CZDS (a particular case of Problem 66), where H denotes the set of Hermite polynomials $\{H_k(x)\}_{k=0}^{\infty}$. Since this topic, to our knowledge, has never been addressed in the literature, all of the results of this section are apparently new. We begin with the relevant definitions.

Definition 97. Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. The *T_H-operator corresponding to the sequence* $\{\gamma_k\}_{k=0}^{\infty}$ is the linear operator on $\mathbb{R}[x]$ which is defined by

$$T_H[H_n(x)] = \gamma_n H_n(x) \qquad (n = 0, 1, 2, \dots).$$

Definition 98. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called an *Hermite Complex Zero Decreasing Sequence* or, for brevity, an *H-CZDS* (which we will also use in the plural), if the T_H-operator corresponding to the sequence $\{\gamma_k\}_{k=0}^{\infty}$ has the property that, for any real polynomial $p(x)$,

$$Z_C\big(T_H[p(x)]\big) \leq Z_C\big(p(x)\big),$$

where $Z_C(f(x))$ is defined as in Notation 56.

Let us begin with the most elementary example of an H-CZDS.

Example 99. Any linear combination (over $\mathbb{R}$) of consecutive Hermite polynomials has only real zeros. Indeed, let α and β be any real numbers. Then, by the Hermite-Poulain Theorem (Theorem 37), the relation $H_n'(x) = 2nH_{n-1}(x)$, and the fact that $H_n(x)$ has only real zeros,

$$\alpha H_n(x) + \beta H_{n-1}(x) = \left(\alpha + \frac{\beta}{2n} D\right)[H_n(x)] \in \mathcal{L} - \mathcal{P}.$$

Therefore, any sequence of the form

$$\{0, 0, 0, \dots, 0, \gamma_n, \gamma_{n+1}, 0, 0, 0, \dots\} \qquad (\gamma_n, \gamma_{n+1} \in \mathbb{R}). \tag{4.20}$$

is an H-CZDS.

Definition 100. We will call a sequence of the form (4.20) a *trivial H-CZDS*.

The existence of non-trivial H-CZDS is the consequence of the following generalization of Turán's theorem (Theorem 49), which can be thought of as a partial analogue of Laguerre's theorem (Theorem 60) for Hermite expansions.

Theorem 101. *Suppose* $p(x) = \sum_{k=0}^{n} a_k H_k(x)$ *is a real polynomial. If* $\varphi(x) \in \mathcal{L} - \mathcal{P}^+$, *then*

$$Z_C\left(\sum_{k=0}^{n} a_k \varphi(k) H_k(x)\right) \leq Z_C\left(\sum_{k=0}^{n} a_k H_k(x)\right).$$

I.e., the sequence $\{\varphi(k)\}_{k=0}^{\infty}$ *is an* H*-CZDS.*

Proof. We first consider the case $\varphi(x) = \alpha + x$, where $\alpha \geq 0$. Let T_H be the T_H-operator corresponding to the sequence $\{\varphi(k)\}_{k=0}^{\infty} = \{\alpha + k\}_{k=0}^{\infty}$ and let

$$\delta = xD - \frac{1}{2}D^2.$$

Then, from Hermite's differential equation (2.7), we have

$$\delta H_n(x) = nH_n(x) \qquad (n = 0, 1, 2, \dots).$$

Thus, for the real polynomial $p(x) = \sum_{k=0}^{n} a_k H_k(x)$, we have

$$\begin{aligned} T_H[p(x)] &= \sum_{k=0}^{n} a_k(\alpha + k)H_k(x) \\ &= \sum_{k=0}^{n} a_k(\alpha I + \delta)H_k(x) \\ &= (\alpha I + \delta)\sum_{k=0}^{n} a_k H_k(x) \\ &= \left(\alpha I + xD - \frac{1}{2}D^2\right) p(x). \end{aligned}$$

By Proposition 68 and our assumption $\alpha \geq 0$, the operator $\alpha + xD - 1/2D^2$ is a complex zero decreasing operator. Therefore

$$Z_C\big(T_H[p(x)]\big) \le Z_C\big(p(x)\big).$$

Now suppose $\varphi(x) \in \mathcal{L}-\mathcal{P}^+$ is a polynomial of degree $m \ge 2$. Then $\varphi(x)$ has only real zeros and non-negative Taylor coefficients, i.e.,

$$\varphi(x) = c\prod_{k=1}^{m}(x+\alpha_k),$$

where $c > 0$ and $\alpha_k \ge 0$. Therefore, the result follows from iteration of the previous case.

Finally, suppose $\varphi(x)$ is a transcendental entire function in the class $\mathcal{L}-\mathcal{P}^+$. Then there is a sequence of polynomials $\{\varphi_j(x)\}_{j=0}^{\infty} \subseteq \mathcal{L}-\mathcal{P}^+$ which converge uniformly on compact subsets to $\varphi(x)$. Let T_{j_H} denote the T_H-operator corresponding to the sequence $\{\varphi_j(k)\}_{k=0}^{\infty}$. Then, by the preceding results, the real polynomial $p(x)$ satisfies

$$Z_C\big(T_{j_H}[p(x)]\big) \le Z_C\big(p(x)\big) \qquad (j = 0, 1, 2, \dots).$$

Since the sequence of polynomials $\{T_{j_H}[p(x)]\}_{j=0}^{\infty}$ converge uniformly on compact subsets to $T_H[p(x)]$, we have, by Hurwitz' theorem, that

$$Z_C\big(T_H[p(x)]\big) \le Z_C\big(p(x)\big).$$

$\square$

Remark 102. We stated that Theorem 101 is a *partial* analogue of Laguerre's theorem (Theorem 60). As the following example shows, we cannot extend Laguerre's theorem in its full generality to Hermite expansions.

Example 103. The sequence $\{1/k!\}_{k=0}^{\infty}$ is, by Laguerre's theorem, a CZDS (see Example 61). However, this sequence is *not* an H-CZDS. Indeed, if we let T_H denote the T_H-operator associated with the sequence $\{1/k!\}_{k=0}^{\infty}$, then

$$p(x) = \frac{1}{2}H_0(x) + \frac{1}{4}H_2(x) = \frac{1}{2}(1) + \frac{1}{4}(4x^2 - 2) = x^2$$

does not have any non-real zeros, while

$$T_H[p(x)] = \frac{1}{2} \cdot \frac{1}{0!}H_0(x) + \frac{1}{4} \cdot \frac{1}{2!}H_2(x) = \frac{1}{2}(1) + \frac{1}{8}(4x^2 - 2) = \frac{1}{2}x^2 + \frac{1}{4}$$

has two non-real zeros.

The next result is the Hermite expansion analogue of a proposition due to Craven and Csordas [11, Proposition 2.2].

Theorem 104. *For every positive integer m the sequence*

$$\left\{k(k-1)(k-2)\cdots(k-m+1)\prod_{i=1}^{p}(k-b_i)\right\}_{k=0}^{\infty} \qquad (b_i \leq m;\ i = 1, 2, 3, \ldots, p) \tag{4.21}$$

is an H-CZDS.

Proof. Let T_H be the T_H-operator associated with the sequence (4.21) and let

$$\delta = xD - \frac{1}{2}D^2.$$

Again, we note that, by Hermite's Differential Equation (2.7),

$$\delta[H_n(x)] = nH_n(x) \qquad (n = 0, 1, 2, \ldots).$$

Thus, for any real polynomial $p(x)$,

$$T_H[p(x)] = \delta(\delta-1)(\delta-2)\cdots\big(\delta-(m-1)\big)\prod_{i=1}^{p}(\delta-b_i)[p(x)],$$

and, by Lemma 27,

$$T_H[p(x)] = \left(x-\frac{1}{2}D\right)^m \left(\prod_{i=1}^{p}\left((m-b_i)+xD-\frac{1}{2}D^2\right)\right) D^m[p(x)].$$

By Rolle's theorem, D^m is a complex zero decreasing operator. Similarly, for each i, the assumption $b_i \le m$ implies $m - b_i \ge 0$. Thus the operator $(m - b_i + xD - 1/2\,D^2)$ is, by Proposition 68, a complex zero decreasing operator. Finally, the operator $(x - 1/2\,D)$ is, by Proposition 67, a complex zero decreasing operator. Therefore,

$$Z_C\big(T_H[p(x)]\big) \le Z_C\big(p(x)\big).$$

□

Remark 105. Just as in Example 62, it can be shown that, for any $m \ge 2$, the sequence (4.21) cannot be interpolated by a function in $\mathcal{L}-\mathcal{P}^+$. Indeed, if there were such a function $\varphi(x) = \sum_{k=0}^{\infty}\alpha_k x^k \in \mathcal{L}-\mathcal{P}^+$ then the fact that $\varphi(1) = 0$ forces all of the Taylor coefficients α_k to be zero, contradicting the fact that $\varphi(m+1) \ne 0$.

Let us combine the preceding results into the following theorem which gives the form of the most general H-CZDS known up to this point.

Theorem 106. *For any function $\varphi(x) \in \mathcal{L}-\mathcal{P}^+$ and any non-negative integer m, the sequence*

$$\left\{\varphi(k)\left(k(k-1)\cdots(k-m+1)\prod_{i=1}^{p}(k-b_i)\right)\right\}_{k=0}^{\infty} \qquad (b_i \le m;\ i = 1,2,3,\ldots,p) \tag{4.22}$$

is an Hermite complex zero decreasing sequence.

Proof. The result is an immediate consequence of Theorem 101, Theorem 104, and the fact that the Hadamard product of two H-CZDS is again an H-CZDS. I.e., if $\{\gamma_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$ are two H-CZDS, then the sequence $\{\gamma_k \cdot \lambda_k\}_{k=0}^{\infty}$ is also an H-CZDS. Indeed, given any real polynomial $p(x) = \displaystyle\sum_{k=0}^{n} a_k H_k(x)$, we have

$$Z_C\left(\sum_{k=0}^{n} a_k H_k(x)\right) \leq Z_C\left(\sum_{k=0}^{n} a_k \gamma_k H_k(x)\right) \leq Z_C\left(\sum_{k=0}^{n} a_k \gamma_k \lambda_k H_k(x)\right).$$

□

Example 103 showed that there are CZDS which are not H-CZDS. However, each H-CZDS we have discovered thus far *is* a CZDS. We will prove that this is true in general. To do so, we will need two technical lemmas which, together, give credence to another Heuristic principle related to the operator $\exp(-D^2/4)$. We have seen that the operator $\exp(-D^2/4)$ tends to attract the non-real zeros of a polynomial $p(x)$ to the real axis (Theorem 78) and that the following inequality always holds (Theorem 38).

$$Z_C\big(\exp(-D^2/4)[p(x)]\big) \leq Z_C(p(x)). \tag{4.23}$$

However, if the imaginary parts of the non-real zeros of the given polynomial $p(x)$ are sufficiently large (how large generally depends on the degree of $p(x)$), then the number of non-real zeros will not decrease, but remain the same, i.e., we will have equality in (4.23). The following two lemmas together demonstrate this principle by showing that if one moves the non-real zeros of a polynomial away from the real axis via the transformation $x \mapsto x/j$ (where j is sufficiently large), then application of the operator $\exp(-D^2/4)$ does not change the number of non-real zeros of the resulting polynomial.

Lemma 107. *Let* $p(x) = \sum_{k=0}^{n} a_k x^k$ *be a real polynomial and, for any non-zero real constant* j*, define*

$$q_j(x) = \exp\left(-\frac{D^2}{4j^2}\right)[p(x)] \qquad \left(D = \frac{d}{dx}\right). \tag{4.24}$$

Then

$$q_j\left(\frac{x}{j}\right) = \sum_{k=0}^{n} \frac{a_k}{(2j)^k} H_k(x) = \exp(-D^2/4)\left[p\left(\frac{x}{j}\right)\right] \qquad \left(D = \frac{d}{dx}\right). \tag{4.25}$$

Proof. For the convenience of the reader, we will recall here the main tool, which is Equation (3.15) of Example 30.

$$\exp\left(-\frac{\alpha}{2}D^2\right)[x^n] = \left(\frac{\alpha}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2\alpha}}\right) \qquad \left(\alpha \neq 0,\ D = \frac{d}{dx}\right). \tag{4.26}$$

Examining the left hand side of (4.25), we have

$$q_j\left(\frac{x}{j}\right) = \left[\exp\left(-\frac{D^2}{4j^2}\right)[p(w)]\right]_{w=x/j} \qquad \left(D = \frac{d}{dw}\right)$$

which, by (4.26), becomes

$$q_j\left(\frac{x}{j}\right) = \left[\sum_{k=0}^{n} a_k \left(\frac{1}{2j}\right)^k H_k(jw)\right]_{w=x/j} = \sum_{k=0}^{n} a_k \left(\frac{1}{2j}\right)^k H_k(x).$$

Similarly, examining the right hand side of (4.25), we have

$$\exp\left(-\frac{D^2}{4}\right)\left[p\left(\frac{x}{j}\right)\right] = \exp\left(-\frac{D^2}{4}\right)\left[\sum_{k=0}^{n} a_k \left(\frac{x}{j}\right)^k\right] \qquad \left(D = \frac{d}{dx}\right)$$

which, by (4.26), becomes

$$\exp\left(-\frac{D^2}{4}\right)\left[p\left(\frac{x}{j}\right)\right] = \sum_{k=0}^{n} a_k \left(\frac{1}{2j}\right)^k H_k(x) \qquad \left(D = \frac{d}{dx}\right).$$

Thus, equation (4.25) holds as claimed. □

We will now take j to be any non-zero natural number in Lemma 107 to obtain a sequence of functions which have a desirable property.

Lemma 108. *Let* $p(x) = \sum_{k=0}^{n} a_k x^k$ *be a real polynomial and define*

$$q_j(x) = \exp\left(-\frac{D^2}{4j^2}\right)[p(x)] \qquad \left(D = \frac{d}{dx};\ j = 1, 2, 3, \ldots\right).$$

Then there exists a natural number N *such that*

$$Z_C\left(q_j\left(\frac{x}{j}\right)\right) = Z_C\big(p(x)\big)$$

whenever $j \geq N$.

Proof. By Lemma 107 and the fact that the operator $\exp(-D^2/4)$ is a complex zero decreasing operator (see Theorem 38),

$$Z_C\left(q_j\left(\frac{x}{j}\right)\right) = Z_C\left(\exp\left(-\frac{D^2}{4}\right)\left[p\left(\frac{x}{j}\right)\right]\right) \leq Z_C\left(p\left(\frac{x}{j}\right)\right) = Z_C\big(p(x)\big). \quad (4.27)$$

Since the coefficients of

$$q_j(x) = \exp\left(-\frac{D^2}{4j^2}\right)[p(x)] = \sum_{k=0}^{[n/2]} \left(-\frac{1}{4j^2}\right)^k \frac{1}{k!} p^{(2k)}(x)$$

tend to the coefficients of $p(x)$ as $j \to \infty$, the polynomials $\{q_j(x)\}_{j=1}^{\infty}$ converge uniformly on compact subsets of $\mathbb{C}$ to $p(x)$. Therefore, by Hurwitz' theorem, there exists an integer N such that

$$Z_C\big(p(x)\big) \leq Z_C\big(q_j(x)\big) = Z_C\left(q_j\left(\frac{x}{j}\right)\right)$$

whenever $j \geq N$. Comparing with (4.27) yields the result. $\square$

We are now in a position to prove that every Hermite complex zero decreasing sequence is a classical complex zero decreasing sequence.

Proposition 109. *If the sequence $\{\gamma_k\}$ is an H-CZDS, then $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS.*

Proof. By way of contradiction, suppose the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is an H-CZDS which is not a classical CZDS. Then there exists a polynomial $f(x) = \displaystyle\sum_{k=0}^{n} a_k x^k$ which has the property that

$$Z_C\big(p(x)\big) > Z_C\big(f(x)\big), \tag{4.28}$$

where $p(x) = \displaystyle\sum_{k=0}^{n} a_k \gamma_k x^k$. By Lemma 108, there exists an integer N such that

$$Z_C\left(q_N\left(\frac{x}{N}\right)\right) = Z_C\big(p(x)\big), \tag{4.29}$$

where

$$q_N(x) = \exp\left(-\frac{D^2}{4N^2}\right)[p(x)].$$

Since the operator $\exp(-D^2/4)$ is a complex zero decreasing operator,

$$Z_C\big(f(x)\big) = Z_C\left(f\left(\frac{x}{N}\right)\right) \geq Z_C\left(\exp(-D^2/4)\left[f\left(\frac{x}{N}\right)\right]\right). \tag{4.30}$$

By equation (4.25) of Lemma 107 and our assumption that $\{\gamma_k\}_{k=0}^{\infty}$ is an H-CZDS,

$$\begin{aligned} Z_C\left(\exp(-D^2/4)\left[f\left(\frac{x}{N}\right)\right]\right) &= Z_C\left(\sum_{k=0}^{n} \frac{a_k}{(2N)^k} H_k(x)\right) \\ &\geq Z_C\left(\sum_{k=0}^{n} \frac{a_k}{(2N)^k}\gamma_k H_k(x)\right) \qquad (4.31) \\ &= Z_C\left(q_N\left(\frac{x}{N}\right)\right). \end{aligned}$$

Thus, by (4.29), (4.30), and (4.31) we have $Z_C\big(f(x)\big) \geq Z_C\big(p(x)\big)$, which contradicts (4.28). $\square$

Remark 110. Before the end of this dissertation, we shall come across another proof of Proposition 109 which does not require the use of the operator $\exp(-D^2/4)$. This method will then be used to show that, remarkably, every Q-CZDS must be a CZDS, *regardless of the choice of the basis Q*. However, we shall postpone this discussion until we arrive at a more suitable setting.

With the aid of Proposition 109, we may now prove the Hermite expansion analogue of the characterization of H-CZDS which can be interpolated by a polynomial (Theorem 63).

Theorem 111. *Let $h(x)$ be a real polynomial. Then $\{h(k)\}_{k=0}^{\infty}$ is an H-CZDS if and only if either*

1. *$h(0) \neq 0$ and $h(x)$ has only real negative zeros, or*

2. *$h(0) = 0$ and $h(x)$ is of the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{k=1}^{p}(x-b_k) \tag{4.32}$$

where $m \geq 1$ and $p \geq 0$ are integers and $b_k < m$ for $k = 1, 2, 3, \ldots, p$.

Proof. Suppose $\{h(k)\}_{k=0}^{\infty}$ is an H-CZDS. Then, by Proposition 109, $\{h(k)\}_{k=0}^{\infty}$ is a CZDS and the result is an immediate consequence of Theorem 63.

Conversely, if $h(x)$ has only real negative zeros, then, by Theorem 101, $\{h(k)\}_{k=0}^{\infty}$ is an H-CZDS. If $h(x)$ is of the form (4.32), then, by Theorem 104, $\{h(k)\}_{k=0}^{\infty}$ is an H-CZDS. $\square$

4.4 Hermite Multiplier Sequences

In this section we will return to the problem of finding an Hermite expansion analogue of Pólya and Schur's characterization of multiplier sequences (see Problem 50). In analogy to the concept of a multiplier sequence, we make the following definition.

Definition 112. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called an *Hermite multiplier sequence* or, for brevity, an H-multiplier sequence if it has the property that its corresponding T_H-operator preserves reality of zeros. I.e., the sequence $\{\gamma_k\}_{k=0}^{\infty}$ has the property that

$$\sum_{k=0}^{n} a_k \gamma_k H_k(x) \in \mathcal{L}-\mathcal{P} \quad \text{whenever} \quad \sum_{k=0}^{n} a_k H_k(x) \in \mathcal{L}-\mathcal{P} \qquad (a_k \in \mathbb{R}).$$

By Turán's theorem (Theorem 49), any sequence which can be interpolated by a real polynomial having only real zeros is an H-multiplier sequence. Furthermore, the generalization of Bleecker and Csordas [1, Theorem 2.7] states that any sequence which can be interpolated by a function in the class $\mathcal{L}-\mathcal{P}^+$ is an H-multiplier sequence. As far as we know, these (i.e., sequences which can be interpolated by functions in $\mathcal{L}-\mathcal{P}^+$) are the only H-multiplier sequences which have been previously discovered.

That we have already discovered some new H-multiplier sequences is a consequence of the results of the previous section and the following lemma.

Lemma 113. *If $\{\gamma_k\}_{k=0}^{\infty}$ is an H-CZDS, then $\{\gamma_k\}_{k=0}^{\infty}$ is an H-multiplier sequence.*

Proof. We first note that a real polynomial $q(x)$ belongs to the class $\mathcal{L}-\mathcal{P}$ if and only if $Z_C\big(q(x)\big) = 0$. Let T_H be the T_H-operator associated with the H-CZDS $\{\gamma_k\}_{k=0}^{\infty}$. If $p(x)$ is a real polynomial which belongs to the class $\mathcal{L}-\mathcal{P}$, then

$$Z_C\big(T_H[p(x)]\big) \leq Z_C\big(p(x)\big) = 0,$$

Thus $Z_C\big(T[p(x)]\big) = 0$, which implies $T[p(x)] \in \mathcal{L}-\mathcal{P}$. Therefore $\{\gamma_k\}_{k=0}^{\infty}$ is an H-multiplier sequence. $\square$

Example 114. By Example 99, any sequence of the form

$$\{0, 0, 0, \dots, 0, \gamma_n, \gamma_{n+1}, 0, 0, 0, \dots\} \qquad (\gamma_n, \gamma_{n+1} \in \mathbb{R}). \tag{4.33}$$

is an H-CZDS and, therefore, is also an H-multiplier sequence.

Definition 115. We will call a sequence of the form (4.33) a *trivial H-multiplier sequence.*

Proposition 116. *For any function $\varphi(x) \in \mathcal{L}-\mathcal{P}^{+}$ and any non-negative integer m, the sequence*

$$\left\{\varphi(k)\left(k(k-1)\cdots(k-m+1)\prod_{i=1}^{p}(k-b_i)\right)\right\}_{k=0}^{\infty} \qquad (b_i \leq m;\ i = 1, 2, 3, \dots, p) \tag{4.34}$$

is an Hermite multiplier sequence.

Proof. The sequence (4.34) is, by Theorem 106 an H-CZDS. Therefore, by Lemma 113, it is also an H-multiplier sequence. $\square$

Remark 117. Since, for $m \geq 2$, it is easy to see that the sequences of the form (4.34) cannot be interpolated by any function in the class $\mathcal{L} - \mathcal{P}^+$ (see Remark 105), these Hermite multiplier sequences are indeed different from those which were already known.

By Proposition 109, every H-CZDS is also a CZDS. As it turns out, we can adapt this result to our current setting.

Proposition 118. *If $\{\gamma_k\}_{k=0}^{\infty}$ is an H-multiplier sequence, then it is a (classical) multiplier sequence.*

Proof. The proof is the same as that of Proposition 109, except we choose the polynomial $f(x)$ so that, in equation (4.28), we have $Z_C\big(f(x)\big) = 0$. $\square$

By Proposition 118, most of the properties of multiplier sequences given in Proposition 45 carry over to H-multiplier sequences. However, we will see that the converse of Proposition 118 is false, i.e., there are classical multiplier sequences which are not H-multiplier sequences. Therefore, some justification is required for some of the following assertions.

Proposition 119. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be an H-multiplier sequence. Then*

(1) *If there exists an integer $m \geq 0$ such that $\gamma_m \neq 0$ and an integer $n > m$ such that $\gamma_n = 0$, then $\gamma_k = 0$ for all $k \geq n$.*

(2) *The elements of* $\{\gamma_k\}_{k=0}^{\infty}$ *are either all of the same sign, or they alternate in sign.*

(3) *The sequence* $\left\{(-1)^k\gamma_k\right\}_{k=0}^{\infty}$ *is also an* H*-multiplier sequence.*

(4) *For any* $r \in \mathbb{R}$*, the sequence* $\{r\gamma_k\}_{k=0}^{\infty}$ *is also an* H*-multiplier sequence.*

(5) *The elements of* $\{\gamma_k\}_{k=0}^{\infty}$ *satisfy Turán's inequality*

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \qquad (k = 1, 2, 3, \dots). \tag{4.35}$$

Proof. By Proposition 118, the H-multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$ is also a multiplier sequence. Therefore, properties (1), (2), and (5) follow from Proposition 45.

Property (3) follows from the relation $H_n(-x) = (-1)^n H_n(x)$. Indeed, if

$$p(x) = \sum_{k=0}^{n} a_k H_k(x)$$

is a real polynomial with only real zeros, then

$$f(x) = T_H[p(x)] = \sum_{k=0}^{n} a_k \gamma_k H_k(x) \in \mathcal{L} - \mathcal{P},$$

where T_H is the T_H-operator associated with the sequence $\{\gamma_k\}_{k=0}^{\infty}$. Therefore,

$$f(-x) = \sum_{k=0}^{n} a_k \gamma_k H_k(-x) = \sum_{k=0}^{n} a_k (-1)^k \gamma_k H_k(x) \in \mathcal{L} - \mathcal{P}.$$

Property (4) is trivial. Indeed, if

$$p(x) = \sum_{k=0}^{n} a_k H_k(x)$$

is a real polynomial with only real zeros, then

$$\sum_{k=0}^{n} a_k r \gamma_k H_k(x) = r \sum_{k=0}^{n} a_k \gamma_k H_k(x) = r T_H[p(x)] \in \mathcal{L} - \mathcal{P}.$$

$\square$

Remark 120. Suppose $\{\gamma_k\}_{k=0}^{\infty}$ is an H-multiplier sequence. Then, by properties (3) and (4) of Proposition 119, the sequences $\{-\gamma_k\}_{k=0}^{\infty}$, $\left\{(-1)^k\gamma_k\right\}_{k=0}^{\infty}$, and $\left\{(-1)^{k+1}\gamma_k\right\}_{k=0}^{\infty}$ are also H-multiplier sequences. Furthermore, by property (2) of Proposition 119, one of these sequences consists entirely of non-negative elements. In what follows, we will often restrict our attention to non-negative H-multiplier sequences. One should keep in mind that analogous results hold in the general case.

In the previous section, we saw that the sequence $\{1/k!\}_{k=0}^{\infty}$ is not an H-CZDS (see Example 103). The same example used to show this can be used to see that $\{1/k!\}_{k=0}^{\infty}$ is not an H-multiplier sequence either. The goal of the remainder of this section is to show that, in general, any non-trivial H-multiplier sequence whose elements are non-negative must be non-decreasing. First, we will demonstrate several preparatory results which are of interest in their own right.

Lemma 121. *Let $n \geq 4$ be an integer and suppose x_1 and x_2 are consecutive zeros of $H_n(x)$ of the same sign. Then $H_{n-2}(x_1)H_{n-2}(x_2) < 0$.*

Proof. From the pure recurrence relation (see equation (2.6))

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)$$

and the fact that x_1 and x_2 are zeros of $H_n(x)$, we have the two equations

$$\begin{aligned} 0 = H_n(x_1) = 2x_1H_{n-1}(x_1) - 2(n-1)H_{n-2}(x_1) \\ 0 = H_n(x_2) = 2x_2H_{n-1}(x_2) - 2(n-1)H_{n-2}(x_2). \end{aligned}$$

Thus

$$H_{n-2}(x_1)H_{n-2}(x_2) = \frac{x_1}{n-1}H_{n-1}(x_1)\frac{x_2}{n-1}H_{n-1}(x_2). \tag{4.36}$$

By the relation $\bigl($see equation (2.5)$\bigr)$

$$H_n'(x) = 2nH_{n-1}(x),$$

equation (4.36) becomes

$$H_{n-2}(x_1)H_{n-2}(x_2) = \frac{x_1 x_2}{4n^2(n-1)^2}H_n'(x_1)H_n'(x_2). \tag{4.37}$$

Furthermore, since the zeros of $H_n(x)$ are all simple and x_1 and x_2 are consecutive zeros of $H_n(x)$, we have $H_n'(x_1)H_n'(x_2) < 0$. This fact, together with equation (4.37) and the fact that we have chosen x_1 and x_2 to be the same sign, shows that $H_{n-2}(x_1)H_{n-2}(x_2) < 0$. □

Remark 122. In particular, Lemma 121 implies that there is a zero of $H_{n-2}(x)$ between every pair of consecutive zeros of $H_n(x)$. In light of the fact that $H_{n-2}(x)$ is a scalar multiple of the second derivative of $H_n(x)$, this is somewhat remarkable. In general, the zeros of the second derivative of a polynomial are not always so well behaved. Consider, for example, the (even) polynomial

$$q(x) = (x^2-1)(x^2-4) = x^4 - 5x^2 + 4.$$

The second derivative of $q(x)$ is $q''(x) = 2(6x^2-5)$ which does not have any zeros in the interval $(1, 2)$.

The next two technical lemmas will also be needed in what follows.

Lemma 123. *Let $a > 0$ and suppose the function $f(x)$ is differentiable on the interval $[0, a]$. If $f(x)$ has a simple zero at $x = 0$, is non-zero at $x = a$, and has exactly m zeros (counting multiplicities) in the interval $(0, a)$, then $(-1)^m f'(0)f(a) > 0$.*

Proof. Let x_0 denote the smallest zero of $f(x)$ in the interval $(0, a)$ (if $f(x)$ has no zeros in this interval, set $x_0 = a$). Then

$$(-1)^m f(x)f(a) > 0 \quad \text{for all} \quad x \in (0, x_0).$$

Therefore, by the definition of the derivative and the fact that $f(0) = 0$,

$$(-1)^m f'(0)f(a) = (-1)^m \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} f(a) = \lim_{x \to 0^+} \frac{(-1)^m f(x)f(a)}{x} \geq 0.$$

By assumption, $f'(0)f(a) \neq 0$, thus $(-1)^m f'(0)f(a) > 0$. $\square$

Lemma 124. *Let $n \geq 3$ be an odd natural number and let a denote the smallest positive zero of $H_n(x)$. Then $H_n(x)$ has constant sign on the interval $(0, a)$ and, for any $x_0 \in (0, a)$,*

$$\operatorname{sign}\big(H_n(x_0)\big) = \operatorname{sign}\big(H_{n-1}(0)\big).$$

Proof. Since n is odd, $H_n(0) = 0$. Since the zeros of $H_{n-1}(x)$ are all simple and separated by the zeros of $H_n(x)$, we have

$$H_{n-1}(0)H_{n-1}(a) < 0. \tag{4.38}$$

From the relation $\big($see (2.5)$\big)$

$$H_n'(x) = 2nH_{n-1}(x)$$

we have

$$H_{n-1}(a) = \frac{1}{2n}H_n'(a) = \frac{1}{2n}\lim_{x\to a}\frac{H_n(x)-H_n(a)}{x-a} = \frac{1}{2n}\lim_{x\to a^-}\frac{H_n(x)}{x-a}.$$

Therefore, the sign of $H_{n-1}(a)$ is opposite of the sign of $H_n(x)$ on the interval $(0, a)$. By inequality (4.38), the sign of $H_{n-1}(0)$ is the same as the sign of $H_n(x)$ on the interval $(0, a)$, which is what was to be shown. □

Proposition 125. *Let $n \geq 2$ be an integer and suppose $\beta \in \mathbb{R}$. Then the polynomial*

$$f_{n,\beta}(x) = H_n(x) + \beta H_{n-2}(x)$$

has only real zeros if and only if $\beta \leq 4\left[\frac{n+1}{2}\right] - 2$.

Proof. Let us first consider the cases $n = 2$ and $n = 3$. From the equation

$$f_{2,\beta}(x) = H_2(x) + \beta H_0(x) = 4x^2 - 2 + \beta,$$

we see that $f_{2,\beta}(x)$ has only real zeros if and only if $\beta \leq 2 = 4\left[\frac{2+1}{2}\right] - 2$. Similarly, from the equation

$$f_{3,\beta}(x) = H_3(x) + \beta H_1(x) = 8x^3 + (2\beta - 12)x,$$

we see that $f_{3,\beta}(x)$ has only real zeros if and only if $\beta \leq 6 = 4\left[\frac{3+1}{2}\right] - 2$.

Now suppose $n \geq 4$ and $\beta \leq 0$. In this case, the polynomial

$$2^n x^n + \beta 2^{n-2}x^{n-2} = (2x)^{n-2}(4x^2 - \beta)$$

has only real zeros. Thus, since the operator $\exp(-D^2/4)$ preserves reality of zeros,

$$\exp(-D^2/4)\left[2^n x^n + \beta 2^{n-2}x^{n-2}\right] = H_n(x) + \beta H_{n-2}(x) = f_{n,\beta}(x)$$

has only real zeros.

Finally we consider the case where $n \geq 4$ and $\beta > 0$. Since the Hermite polynomials satisfy the relation $H_k(-x) = (-1)^k H_k(x)$, we have

$$f_{n,\beta}(-x) = (-1)^n f_{n,\beta}(x). \tag{4.39}$$

In light of this fact, we will count the number of zeros of $f_{n,\beta}(x)$ on the positive real axis. Let

$$0 < x_1 < x_2 < \cdots < x_m \qquad \left(m = \left[\frac{n}{2}\right]\right)$$

denote the positive zeros of $H_n(x)$. By Lemma 121,

$$f_{n,\beta}(x_i) f_{n,\beta}(x_{i+1}) = \beta^2 H_{n-2}(x_i) H_{n-2}(x_{i+1}) < 0 \qquad (i = 1, 2, 3, \ldots, m-1).$$

Therefore, for each $i = 1, 2, 3, \ldots, m-1$, the polynomial $f_{n,\beta}(x)$ has an odd number of zeros in the interval (x_i, x_{i+1}). We claim that $f_{n,\beta}(x)$ has exactly one zero (counting multiplicities) in each of these intervals. Indeed, if this were not the case, then $f_{n,\beta}(x)$ would have at least $(m-1)+2 = m+1$ positive zeros. By equation (4.39), $f_{n,\beta}(x)$ would then have at least $2m+2$ real zeros. But

$$2m + 2 = 2\left[\frac{n}{2}\right] + 2 \geq n + 1$$

and, since $\deg\big(f_{n,\beta}(x)\big) = n$, we have contradicted the Fundamental Theorem of Algebra. Therefore, $f_{n,\beta}(x)$ has exactly one zero in each of the intervals (x_i, x_{i+1}) as claimed. Furthermore, we have now demonstrated that $f_{n,\beta}(x)$ has at least

$$2m - 2 = 2\left[\frac{n}{2}\right] - 2 = \begin{cases} n-2 & \text{if } n \text{ is even} \\ n-3 & \text{if } n \text{ is odd} \end{cases} \tag{4.40}$$

real zeros, each of which lies in $(-x_m, -x_1) \cup (x_1, x_m)$.

We claim that $f_{n,\beta}(x)$ does not have any zeros in the interval (x_m, ∞). Indeed, $H_n(x)$ and $H_{n-2}(x)$ are both positive on the interval (x_m, ∞) and we have assumed

$\beta > 0$. Whence, $f_{n,\beta}(x)$ is also positive and, therefore, does not vanish on the interval (x_m, ∞). Similarly, $f_{n,\beta}(x)$ does not have any zeros in the interval $(-\infty, -x_m)$.

The question of whether or not $f_{n,\beta}(x)$ has only real zeros now depends entirely on its behavior in the interval $(-x_1, x_1)$. From the explicit formula for the Hermite polynomials (2.3), we have

$$H_k(0) = \begin{cases} \dfrac{(-1)^{k/2}k!}{(k/2)!} & \text{if } k \text{ is even} \\ \\ 0 & \text{if } k \text{ is odd} \end{cases} \tag{4.41}$$

In particular, if $k \geq 2$ is even, then a calculation shows that

$$H_k(0) = -2(k-1)H_{k-2}(0). \tag{4.42}$$

Suppose n is even. In this case we only have two zeros to account for (see equation $\big($4.40$\big)$). Thus $f_{n,\beta}(x)$ has only real zeros if and only if it has a zero in the interval $[0, x_1)$, which occurs if and only if $f(0)f(x_1) \leq 0$. By equation (4.42) and the fact that $H_n(x_1) = 0$,

$$f(0)f(x_1) = \big(H_n(0) + \beta H_{n-2}(0)\big)\big(H_{n-2}(x_1)\big) = \big(-2(n-1) + \beta\big)H_{n-2}(0)H_{n-2}(x_1).$$

Since H_{n-2} does not vanish on $[-x_1, x_1]$, we see that $f_{n,\beta}(0)f_{n,\beta}(x_1) \leq 0$ if and only if $-2(n-1) + \beta \leq 0$. Therefore, $f_{n,\beta}(x)$ has only real zeros if and only if

$$\beta \leq 2(n-1) = 2n - 2 = 4\left[\frac{n+1}{2}\right] - 2.$$

Now suppose n is odd. In this case $f_{n,\beta}(x)$ will have only real zeros if and only if $f_{n,\beta}(x)$ either has a multiple zero at the origin, or has a simple zero at the origin and another simple zero in the interval $(0, x_1)$. In the former case, $f'(0) = 0$ and in the latter, by Lemma 123, $f'(0)f(x_1) < 0$. Thus $f_{n,\beta}(x)$ has only real zeros if and only if $f'(0)f(x_1) \leq 0$. But

$$\begin{aligned} f'(0)f(x_1) &= \left[H_n'(0) + \beta H_{n-2}'(0)\right] H_{n-2}(x_1) \\ &= \left[2nH_{n-1}(0) + \beta 2(n-2)H_{n-3}(0)\right] H_{n-2}(x_1), \end{aligned}$$

which, by equation (4.42), becomes

$$f'(0)f(x_1) = 2(n-2)(\beta - 2n)H_{n-3}(0)H_{n-2}(x_1).$$

Since x_1 lies in the interval $(0, a)$, where a is the smallest positive zero of $H_{n-2}(x)$, we have by, Lemma 124, that $H_{n-3}(0)H_{n-2}(x_1) > 0$. Therefore $f'(0)f(x_1) \leq 0$ if and only if $\beta - 2n \leq 0$. I.e. $f_{n,\beta}(x)$ has only real zeros if and only if

$$\beta \leq 2n = 2(n+1) - 2 = 4\left[\frac{n+1}{2}\right] - 2.$$

□

The usefulness of Proposition 125 lies in the fact that it gives a *necessary and sufficient* condition for reality of zeros of a particular type of Hermite expansion.

Lemma 126. *If the sequence of non-negative real numbers* $\{\gamma_k\}_{k=0}^{\infty}$ *is a non-trivial* H*-multiplier sequence, then there exists an integer* $m \geq 0$ *such that* $\gamma_k = 0$ *for every non-negative integer* $k < m$ *and* $\gamma_k \neq 0$ *for every integer* $k \geq m$*.*

Proof. Since $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial H-multiplier sequence, it has at least one non-zero element (see Definition 115). Set $m = \min\{k : \gamma_k \neq 0\}$. We claim that γ_{m+1} and γ_{m+2} are non-zero. Indeed, if either of them were zero, then, by property (1) of Proposition 119, all subsequent elements of the sequence would also be zero, contradicting the fact that the sequence is a non-trivial H-multiplier sequence.

By way of contradiction, suppose there is an integer $n > m+2$ for which $\gamma_n = 0$. By Proposition 125, we can choose a_m and a_{m+2} so that the polynomial

$$p(x) = a_m \gamma_m H_m(x) + a_{m+2}\gamma_{m+2} H_{m+2}(x) \notin \mathcal{L}-\mathcal{P}.$$

By the theorem of Turán which gives a sufficient condition for reality of zeros of an Hermite expansion (Theorem 84), we may pick a_n large enough so that

$$f(x) = a_m H_m(x) + a_{m+2} H_{m+2}(x) + a_n H_n(x) \in \mathcal{L}-\mathcal{P}.$$

But then $T_H[f(x)] = p(x) \notin \mathcal{L}-\mathcal{P}$, where T_H denotes the T_H-operator corresponding to the sequence $\{\gamma_k\}_{k=0}^{\infty}$. Thus we have contradicted the fact that $\{\gamma_k\}_{k=0}^{\infty}$ is an H-multiplier sequence. Therefore, $a_k \neq 0$ for all $k \geq m$. $\square$

Theorem 127. *If the sequence of non-negative real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial H-multiplier sequence, then $\gamma_k \leq \gamma_{k+1}$ for every integer $k \geq 0$.*

Proof. By Lemma 126, there exists an integer $m \geq 0$ such that $\gamma_k = 0$ for $k < m$ and $\gamma_k \neq 0$ for $k \geq m$. Fix an integer n such that $n \geq m$. Note that, in particular, γ_n, γ_{n+1}, and γ_{n+2} are each non-zero. By Lemma 125, the polynomial

$$f(x) = H_{n+2}(x) + \left(4\left[\frac{n+3}{2}\right] - 2\right) H_n(x)$$

has only real zeros. Let T_H denote the T_H-operator associated with the H-multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$. Then

$$T_H[f(x)] = \gamma_{n+2}\left(H_{n+2}(x) + \left(4\left[\frac{n+3}{2}\right] - 2\right)\frac{\gamma_n}{\gamma_{n+2}} H_n(x)\right)$$

has only real zeros. Thus, by Lemma 125,

$$\left(4\left[\frac{n+3}{2}\right] - 2\right)\frac{\gamma_n}{\gamma_{n+2}} \leq \left(4\left[\frac{n+3}{2}\right] - 2\right). \tag{4.43}$$

Since $\left(4\left[\frac{n+2}{2}\right]-2\right)>0$, it follows from (4.43) and the fact that $\gamma_n \neq 0$ that

$$0 < \frac{\gamma_n}{\gamma_{n+2}} \leq 1. \tag{4.44}$$

By Property 5 of Proposition 119, every H-multiplier sequence satisfies Turán's inequality (4.35). Thus,

$$\gamma_{n+1}^2 - \gamma_n\gamma_{n+2} \geq 0,$$

from which we have

$$\left(\frac{\gamma_{n+1}}{\gamma_n}\right)^2 - \frac{\gamma_{n+2}}{\gamma_n} \geq 0. \tag{4.45}$$

Thus, by the inequalities (4.44) and (4.45),

$$\left(\frac{\gamma_{n+1}}{\gamma_n}\right)^2 \geq \frac{\gamma_{n+2}}{\gamma_n} \geq 1.$$

Therefore $\gamma_n \leq \gamma_{n+1}$. □

Remark 128. Any sequence which is not an H-multiplier sequence cannot be an H-CZDS (see Lemma 113). Thus, any H-CZDS which consists entirely of non-negative elements must be non-decreasing.

Corollary 129. *For any $r > 0$, the sequence $\left\{r^k\right\}_{k=0}^{\infty}$ is an H-CZDS if and only if $r \geq 1$.*

Proof. If $0 < r < 1$, then $\left\{r^k\right\}_{k=0}^{\infty}$ is a sequence of decreasing non-negative numbers. Therefore, by Theorem 127 and Remark 128, $\left\{r^k\right\}_{k=0}^{\infty}$ is not an H-CZDS.

If $r \geq 1$, then the sequence $\left\{r^k\right\}_{k=0}^{\infty}$ can be interpolated by the function

$$r^x = \exp(\ln r^x) = \exp(x \ln r)$$

which, since $\ln r \geq 0$, belongs to the class $\mathcal{L}-\mathcal{P}^{+}$. Therefore, by Proposition 101, $\left\{r^{k}\right\}_{k=0}^{\infty}$ is an H-CZDS. $\square$

So far, every H-multiplier sequence which we have come across is also an H-CZDS. One may be inclined to ask if there are any H-multiplier sequences which are not H-CZDS. Indeed, in the classical setting we have seen that there are multiplier sequences which are not CZDS (see Example 64). In particular, one could ask whether or not the multiplier sequence $\left\{1+k+k^{2}\right\}_{k=0}^{\infty}$ (which is not a CZDS and, therefore, not an H-CZDS) is an H-multiplier sequence. However, none of the methods we have developed thus far yield an answer to this question. In the next chapter, we will see that $\left\{1+k+k^{2}\right\}_{k=0}^{\infty}$ *is* an H-multiplier sequence. In order to prove this, we shall generalize a curve theorem due to Pólya, which itself is a sort of unification of three major results in this area which we have already seen.

Before moving on to the discussion of the curve theorem, we will demonstrate why one is inclined to believe that *every* non-negative multiplier sequence which is non-decreasing must also be an H-multiplier sequence. As it turns out, non-negative multiplier sequences which are non-decreasing enjoy a remarkable geometric property.

Theorem 130. (Gauss-Lucas Theorem [33, p. 84]) *Let K be any convex polygon enclosing all the zeros of the polynomial $f(z)$. Then the zeros of $f'(z)$ lie in K.*

Craven and Csordas examined the Gauss-Lucas Theorem in the context of multiplier sequences, which led them to make the following definition.

Definition 131. ([8, p. 419]) A sequence $\{\gamma_k\}_{k=0}^{\infty}$ of real numbers is said to possess the *Gauss-Lucas property* if it satisfies the following condition. Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be an arbitrary complex polynomial. If K is a convex region containing the origin and all the zeros of $f(z)$, and if the polynomial $g(z) = \sum_{k=0}^{n} a_k \gamma_k z^k$ is not identically zero, then the zeros of $g(z)$ also lie in K.

If the polynomial $f(z)$ in Definition 131 has only real zeros, then we may take K to be a closed interval of real numbers, and we see that the resulting polynomial $g(z)$ will also have only real zeros. Thus, every sequence which possesses the Gauss-Lucas property is also a multiplier sequence. Furthermore, by the Gauss-Lucas Theorem, the sequence $\{k\}_{k=0}^{\infty}$ possesses the Gauss-Lucas property. Craven and Csordas showed that this is true of any non-negative multiplier sequence which is non-decreasing, and, in fact, these are the *only* sequences which posses the Gauss-Lucas property.

Theorem 132. (Craven-Csordas [8, Theorem 2.8]) *The sequence of non-negative real numbers $\{\gamma_k\}_{k=0}^{\infty}$ possesses the Gauss-Lucas property if and only if $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence which satisfies $0 \le \gamma_n \le \gamma_{n+1}$ for all non-negative integers n.*

The next lemma will allow us to relate these results to H-multiplier sequences.

Lemma 133. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and define the linear operators T and T_H on $\mathbb{R}[x]$ by*

$$T[x^n] = \gamma_n x^n \qquad (n = 0, 1, 2, \dots) \tag{4.46}$$

and

$$T_H[H_n(x)] = \gamma_n H_n(x) \qquad (n = 0, 1, 2, \dots),$$

where $H_n(x)$ denotes the n^{th} Hermite polynomial. Then, for any polynomial $p(x)$,

$$T_H[p(x)] = \left(e^{-\frac{D^2}{4}}\, T\, e^{\frac{D^2}{4}}\right)[p(x)].$$

Proof. By relation (3.15), we have

$$e^{-\frac{D^2}{4}}[x^n] = \frac{1}{2^n} H_n(x) \qquad (n = 0, 1, 2, \dots),$$

from which we also obtain

$$e^{\frac{D^2}{4}}[H_n(x)] = (2x)^n \qquad (n = 0, 1, 2, \dots). \tag{4.47}$$

Thus, for any $n = 0, 1, 2, \dots,$ we have

$$\begin{aligned} \left(e^{-\frac{D^2}{4}}\, T\, e^{\frac{D^2}{4}}\right)[H_n(x)] &= \left(e^{-\frac{D^2}{4}}\, T\right)[2^n x^n] \\ &= \left(e^{-\frac{D^2}{4}}\right)[2^n \gamma_n x^n] \\ &= \gamma_n H_n(x). \end{aligned}$$

Therefore the linear operators T_H and $e^{-\frac{D^2}{4}}\, T\, e^{\frac{D^2}{4}}$, since they agree on the basis elements $H_n(x)$ $(n = 0, 1, 2, \dots)$, are equal on all of $\mathbb{R}[x]$. □

Remark 134. Suppose $\{\gamma_k\}_{k=0}^{\infty}$ is a non-negative multiplier sequence which is non-decreasing. Then the following heuristic principle leads us to believe that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ must also be an H-multiplier sequence. If the zeros of $p(x)$ are all real, then the polynomial $f(x) = e^{\frac{D^2}{4}}[p(x)]$ will generally have non-real zeros. By Theorem 132, the zeros of $T[f(x)]$ will lie in the convex hull of the zeros of $f(x)$, where T is defined by (4.46). In particular the imaginary parts of the zeros of $T[f(x)]$ will generally be smaller than those of $f(x)$. Thus the polynomial $T_H[p(x)] = e^{-\frac{D^2}{4}}[T[f(x)]]$ *should* have all its zeros real.

A more precise result is demonstrated by the following proposition.

Proposition 135. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a non-negative multiplier sequence which is non-decreasing and let T_H denote the T_H-operator associated with the sequence $\{\gamma_k\}_{k=0}^{\infty}$. If $p(x)$ is a real polynomial having only real zeros, and if $f(x) = e^{\frac{D^2}{4}}[p(x)]$ has all its zeros in the strip $S(\sqrt{2}) = \left\{z \,:\, \mathrm{Im}\ z \leq \sqrt{2}\right\}$, then the polynomial $T_H[p(x)] \in \mathcal{L}-\mathcal{P}$.*

Proof. Let T be the linear operator on $\mathbb{R}[x]$ defined by $T[x^n] = \gamma_n x^n$ $(n = 0, 1, 2, \dots)$. Then, by Theorem 132, either the polynomial $T[f(x)]$ is identically zero or all of its zeros lie in the strip $S(\sqrt{2})$. Therefore, by Lemma 133 and Corollary 79,

$$T_H[p(x)] = \left(e^{-\frac{D^2}{4}}\, T\, e^{\frac{D^2}{4}}\right)[p(x)] = e^{-\frac{D^2}{4}}\left[T[f(x)]\right] \in \mathcal{L}-\mathcal{P}.$$

□

Unfortunately, if $p(x)$ has only real zeros, then the zeros of $e^{\frac{D^2}{4}}[p(x)]$ need not lie in the strip $S(\sqrt{2})$. Indeed, by the relation (3.15),

$$e^{\frac{D^2}{4}}[x^n] = \frac{(-1)^n}{2^n} H_n(-ix),$$

and the imaginary part of the zeros of $H_n(-ix)$ become arbitrarily large as $n \to \infty$. Therefore, we will need to develop other methods to show that every non-negative multiplier sequence which is non-decreasing must also be an H-multiplier sequence.

Chapter 5
A Curve Theorem

5.1 Pólya's Curve Theorem

In 1916, Pólya discovered a result which he stated as follows.

Theorem 136. (Pólya [25]) *Suppose the real polynomial*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad (a_n \neq 0)$$

has only real zeros and that

$$q(x) = b_0 + b_1 x + \cdots + b_n x^n + b_{n+1} x^{n+1} + \cdots + b_{n+m} x^{n+m} \tag{5.1}$$

$(m \geq 0)$ *also has only real zeros and that the first* $n+1$ *coefficients* $b_0, b_1, b_2, \ldots, b_n$ *are all positive. Then the* n^{th} *order curve*

$$F(x,y) = b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \cdots + b_n x^n f^{(n)}(y) = 0 \tag{5.2}$$

has n *intersection points with each of the lines*

$$sx - ty + u = 0, \tag{5.3}$$

where $s \geq 0$, $t \geq 0$, $s + t > 0$, *and* $u \in \mathbb{R}$.

Figure 1 below demonstrates the graphical representation of the curve $F(x,y) = 0$, where we have chosen $q(x) = (1+x)^4$ and $f(x) = x^3 - x$. In general, by declaring that (5.2) is an n^{th} order curve, it is meant that there are n curves in the plane such that the set $\{(x,y) : F(x,y) = 0\}$ is equal to the union of the images of these curves.

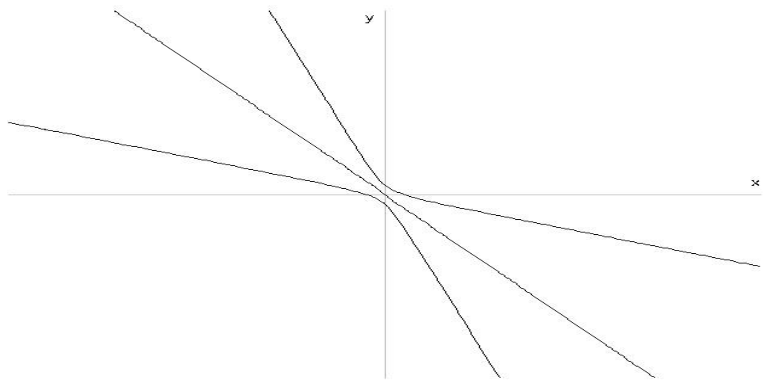

Figure 1: The curve $F(x, y) = 0$ for $\deg(q) > \deg(f)$.

By declaring that a given line has n intersection points with the n^{th}-order curve, it is meant (in this particular setting) that the line intersects each of the n curves exactly once. This clarification is important since, in general, the actual number of points of intersection between the given line and the set $\{(x, y) : F(x, y) = 0\}$ may be less than n. Thus, we must bear in mind that we are counting intersections with some sort of a notion multiplicity. Indeed, by solving for one of the variables x or y in terms of the other in the equation of the line (5.3) and then substituting into the polynomial $F(x, y)$, we obtain a polynomial in one variable. Each real zero of the resulting polynomial in one variable corresponds to an intersection point and, conversely, every intersection point corresponds to a real zero of the resulting polynomial. Thus, the intersection point which corresponds to a zero of multiplicity m should be considered as an intersection point of multiplicity m. These clarifications bring to light the true nature of this theorem. Although the theorem itself is geometric in nature, the underlying theme is that of transformations which preserve reality of zeros.

We also note that, in a more general setting, if $G(x, y)$ is an arbitrary polynomial in two variables x and y and $G(x, y) = 0$ represents an n^{th} order curve, then it is possible that a given line does not have any intersections with one of the n curves. Similarly, it is possible that a given line has several intersections with only one of the n curves.

In the original proof of Theorem 136, Pólya omitted several technical details which demonstrate that these curves do indeed exist. The goal of this chapter is to justify the existence of these curves and to prove another curve theorem which is better suited to polynomials of arbitrary degree. Before getting into this detailed discussion, let us get a better feel for Theorem 136 by examining several important special cases.

In the case where $x = 1$, equation (5.2) reduces to

$$b_0 f(y) + b_1 f'(y) + b_2 f''(y) + \cdots b_n f^{(n)}(y) = 0.$$

Thus Theorem 136 is true in this case, since

$$g(D)f(y) = b_0 f(y) + b_1 f'(y) + b_2 f''(y) + \cdots b_n f^{(n)}(y) \qquad \left(D = \frac{d}{dy}\right),$$

has, by the Hermite-Poulain Theorem (Theorem 37), only real zeros.

Similarly, in the case where $y = 0$, equation (5.2) reduces to

$$a_0 b_0 + 1! a_1 b_1 x + 2! a_2 b_2 x^2 + \cdots + n! a_n b_n = 0,$$

which, by Schur's theorem (Theorem 39), has only real zeros.

It should be noted that it was Pólya's original intention to prove Theorem 136 so that one could obtain the deeper result (Schur's theorem) from the one which is easier to prove (the Hermite-Poulain Theorem). As it turns out, there is another consequence which, in the context of multiplier sequences, is the most interesting. Let us first introduce the following notation which will facilitate the discussion.

Notation 137. For any polynomial

$$q(x) = \sum_{k=0}^{n} a_k x^k$$

we define

$$\widetilde{q}(x) = \sum_{k=0}^{n} a_k k! \binom{x}{k} = a_0 + a_1 x + a_2 x(x-1) + \cdots + a_n \prod_{k=1}^{n} (x - k + 1). \tag{5.4}$$

The usefulness of Notation 137 is demonstrated by the following lemma.

Lemma 138. *For any polynomial* $q(x)$*,*

$$q(x)e^x = \sum_{k=0}^{\infty} \frac{\widetilde{q}(k)}{k!} x^k,$$

where $\widetilde{q}(x)$ *is defined by equation (5.4) of Notation 137.*

Proof. For any integer $k \geq 0$,

$$x^k e^x = \sum_{j=0}^{\infty} \frac{x^{j+k}}{j!} = \sum_{j=k}^{\infty} \frac{x^j}{(j-k)!} = \sum_{j=0}^{\infty} k! \binom{j}{k} \frac{x^j}{j!},$$

where, as usual, we set $\binom{j}{k} = 0$ whenever $j < k$. Therefore, if $q(x) = \sum_{k=0}^{n} a_k x^k$, then

$$q(x)e^x = \sum_{k=0}^{n} a_k x^k e^x = \sum_{k=0}^{n} \sum_{j=0}^{\infty} a_k k! \binom{j}{k} \frac{x^j}{j!} = \sum_{j=0}^{\infty} \sum_{k=0}^{n} a_k k! \binom{j}{k} \frac{x^j}{j!} = \sum_{j=0}^{\infty} \frac{\widetilde{q}(j)}{j!} x^j.$$

□

Remark 139. By the transcendental characterization of multiplier sequences (Theorem 46), if the polynomial $q(x)$ has only real negative zeros, then the sequence

$\{\widetilde{q}(k)\}_{k=0}^{\infty}$ is a multiplier sequence. For example, if we set $q(x) = (1+x)^2$, then $\widetilde{q}(x) = 1 + x + x^2$. Thus, as we saw in Example 48, $\left\{1+k+k^2\right\}_{k=0}^{\infty}$ is a multiplier sequence. Furthermore, in Example 48 we saw that the linear operator T defined by $T[x^n] = (1+n+n^2)x^n$ can be represented as $T = 1 + 2xD + x^2D^2$ which one obtains by replacing x^k by x^kD^k in the Taylor expansion of the original polynomial $q(x)$. This turns out to be true in general.

Lemma 140. *If* $q(x) = \displaystyle\sum_{k=0}^{m} a_k x^k$ *is a real polynomial and* $\widetilde{q}(x)$ *is defined by equation (5.4) of Notation 137, then the linear operator* T *on* $\mathbb{R}[x]$ *defined by* $T[x^n] = \widetilde{q}(n)x^n$ $(n = 0, 1, 2, \dots)$ *can be represented as*

$$T = \sum_{k=0}^{m} a_k x^k D^k.$$

Proof. For any integers $k \geq 0$ and $n \geq 0$,

$$D^k x^n = k!\binom{n}{k}x^{n-k},$$

where we adopt the convention that $\binom{n}{k} = 0$ whenever $k > n$. Therefore, for any integer $n \geq 0$,

$$\left(\sum_{k=0}^{m} a_k x^k D^k\right)[x^n] = \sum_{k=0}^{m} a_k k!\binom{n}{k}x^n = \widetilde{q}(n)x^n.$$

□

Corollary 141. *If* $q(x) = \displaystyle\sum_{k=0}^{m} a_k x^k$ *is a real polynomial which has only real non-positive zeros, then the linear operator*

$$T = \sum_{k=0}^{m} a_k x^k D^k$$

preserves reality of zeros.

Proof. By hypothesis and Lemma 138, we have

$$q(x)e^x = \sum_{k=0}^{\infty} \frac{\widetilde{q}(k)}{k!} x^k \in \mathcal{L} - \mathcal{P}^+.$$

By the transcendental characterization of multiplier sequences (Theorem 46), the sequence $\{\widetilde{q}(k)\}_{k=0}^{\infty}$ is a multiplier sequence, where $\widetilde{q}(x)$ is defined by equation (5.4) of Notation 137. Therefore, by Lemma 140, the linear operator

$$T = \sum_{k=0}^{m} a_k x^k D^k$$

preserves reality of zeros. □

Preparatory remarks aside, we may now see another consequence of Pólya's curve theorem. If we take $y = x$ then equation (5.2) of Theorem 136 reduces to

$$b_0 f(x) + b_1 x f'(x) + b_2 x^2 f''(x) + \cdots + b_n x^n f^{(n)}(x) = 0. \tag{5.5}$$

If we assume that *all* of the coefficients $b_0, b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{n+m}$ of $q(x)$ in equation (5.1) are positive, then the zeros of $q(x)$ are all real and negative and, therefore, by Corollary 141,

$$b_0 f(x) + b_1 x f'(x) + b_2 x^2 f''(x) + \cdots + b_n x^n f^{(n)}(x)$$

has only real zeros. Thus Theorem 136 holds in this special case as well.

Theorem 136 is remarkable in the way it unifies the Hermite-Poulain Theorem, Schur's theorem, and the transcendental characterization of multiplier sequences. However, one should take care to note that Theorem 136 does not yield a new proof of either of these major theorems. The reason for this is that there are significant restrictions on the degrees of the polynomials in question. Indeed, we begin with a polynomial $f(x)$ of fixed degree n and then apply a differential operator of order *greater than or equal to* n. For example, if we wanted to use Theorem 136 to show that $\left\{1+k+k^2\right\}_{k=0}^{\infty}$ is a multiplier sequence, then we would take $q(x) = 1+2x+x^2$. However, we would immediately run into trouble since then we may only apply the theorem to polynomials $f(x)$ of degree $n \leq 2$. In the remainder of this chapter, we shall remedy this shortcoming at the expense of requiring that all the coefficients b_k of $q(x)$ be non-negative. Furthermore, we will prove a more general curve theorem which will be more appropriate to our investigation of Hermite multiplier sequences. In particular, this new tool will provide a way for us to *completely characterize* all Hermite multiplier sequences.

5.2 Existence of Curves and Intersections

We mentioned in the previous section that some details were omitted from the original proof of Pólya's curve theorem (Theorem 136). In particular, it was not clear why the equation $F(x,y) = 0$ $\big($see equation (5.2)$\big)$ should represent an n^{th} order curve. Pólya demonstrated that, as a consequence of the Hermite-Poulain Theorem, each vertical line must have n intersections with the set $\{(x,y) : F(x,y) = 0\}$. However, why we can conclude that there are n curves is not made precise. Similarly, although the intuitive idea behind why the curves must intersect the given line is provided, a rigorous proof is not. The goal of this section is to fill in these details in a more general setting. First we need a sort of continuous analogue of Hurwitz' theorem 11) and some of its consequences (Corollary 12 and Proposition 13).

Proposition 142. *Suppose* $f(z) = \sum_{k=0}^{n} b_k z^k$ $(a_n \neq 0)$ *is a complex polynomial and, for each* $\alpha \in \mathbb{R}$, $g_\alpha(z) = \sum_{k=0}^{m} b_{\alpha,k} z^k$ *is a complex polynomial, where* $m \geq n$ *is a fixed integer. Suppose also that*

$$\lim_{\alpha \to \omega} b_{\alpha,k} = \begin{cases} b_k & \text{for } k = 0, 1, 2, \ldots, n \\ 0 & \text{for } k = n+1, n+2, n+3, \ldots, m \end{cases}$$

where $\omega \in [-\infty, \infty]$. *Then, given any simple closed curve* C *which does not pass through any of the zeros of* $f(z)$, *the polynomials* $g_\alpha(z)$ *and* $f(z)$ *have the same number of zeros inside* C *whenever* α *is sufficiently close to* ω.

Proof. Suppose C is a simple closed curve which does not pass through any of the zeros of $f(z)$. Choose real numbers $\lambda > 0$ and $\nu > 0$ such that,

$$\min_{z \in C} |f(z)| \geq \lambda \qquad \text{and} \qquad \max_{z \in C} |z| \leq \nu.$$

Then, for $z \in C$,

$$\begin{aligned} |g_\alpha(z) - f(z)| &= \left| \sum_{k=0}^{n} (b_{\alpha,k} - b_k) z^k + \sum_{k=n+1}^{m} b_{\alpha,k} z^k \right| \\ &\leq \sum_{k=0}^{n} |b_{\alpha,k} - b_k| |z|^k + \sum_{k=n+1}^{m} |b_{\alpha,k}| |z|^k \\ &\leq \sum_{k=0}^{n} |b_{\alpha,k} - b_k| \nu^k + \sum_{k=n+1}^{m} |b_{\alpha,k}| \nu^k \end{aligned}$$

can be made arbitrarily small for all values of α which are sufficiently close to ω. In particular, for any $z \in C$, $|g_\alpha(z) - f(z)|$ can made to be strictly less than the lower bound λ of $|f(z)|$ on C. Thus, by Rouché's theorem, $g_\alpha(z)$ and $f(z)$ have the same number of zeros inside C whenever α is sufficiently close to ω. $\square$

We can now give sufficient conditions under which we can be assured that an equation of the form $F(x,y)=0$ represents an n^{th} order curve. In fact, our conditions will guarantee that we actually have n continuous *functions* such that the union of the graphs of these functions is equal to the set $\{(x,y) : F(x,y)=0\}$.

Proposition 143. *Let n be a non-negative integer. Suppose $\{q_k(x)\}_{k=0}^{n}$ is a set of real polynomials and that $q_0(x) \neq 0$ for all $x \in \mathbb{R}$. Furthermore, suppose that, for each fixed $x \in \mathbb{R}$, the polynomial*

$$p_x(z) = \sum_{k=0}^{n} q_k(x) z^k$$

has only real zeros. Let $f(y) = \sum_{k=0}^{m} a_k y^k$ $(a_m \neq 0)$ be a real polynomial having only real simple zeros and define

$$F(x,y) = \sum_{k=0}^{n} q_k(x) f^{(k)}(y).$$

Then there is a set of real continuous functions $\{\varphi_k(x)\}_{k=1}^{m}$ such that

$$F(x,y) = q_0(x) a_m \prod_{k=1}^{m} (y - \varphi_k(x)) \tag{5.6}$$

and, for all $x \in \mathbb{R}$,

$$\varphi_1(x) < \varphi_2(x) < \cdots < \varphi_m(x).$$

Proof. First note that, since $q_0(x)$ is non-vanishing on all of $\mathbb{R}$, the polynomial $F(x,y)$ (as a polynomial in y) is of degree m. Since, for each fixed $x \in \mathbb{R}$, $p_x(z)$ has only real zeros and $f(y)$ has only simple real zeros, it follows from the Hermite-Poulain Theorem (Theorem 37) that, for each fixed $x \in \mathbb{R}$, the polynomial $p_x\left(\frac{d}{dy}\right)[f(y)] = F(x,y)$ has m simple real zeros. Therefore, one can define a function $\Phi : \mathbb{R} \to \mathbb{R}^m$ by

$$\Phi(x) = \langle \varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_m(x) \rangle,$$

where $\varphi_1(x) < \varphi_2(x) < \cdots < \varphi_m(x)$ are the distinct zeros of $F_x(y) = F(x,y)$. Furthermore, the representation of $F(x,y)$ in equation (5.6) now follows from the fact that the leading coefficient of $F(x,y)$, as a polynomial in y, is $q_0(x)a_m$.

It remains to show that, for each $k = 1, 2, 3 \ldots, m$, the function $\varphi_k(x)$ is continuous. To this end, fix $a \in \mathbb{R}$ and let $\epsilon > 0$. Let C_1, C_2, $\ldots$, C_m be non-intersecting circles with centers

$$\big(\varphi_1(a), 0\big),\ \big(\varphi_2(a), 0\big),\ \ldots,\ \big(\varphi_m(a), 0\big),$$

respectively, each of which has a radius which is less than ϵ. By Proposition 142, there exists $\delta > 0$ such that the complex polynomial $F_x(z) = F(x,z)$ has the same number of zeros as the complex polynomial $F_a(z) = F(a,z)$ in each of the circles C_i whenever $|x-a| < \delta$. Thus, $|\varphi_i(x) - \varphi_i(a)| < \epsilon$ whenever $|x-a| < \delta$, and each of the functions $\varphi_i(x)$ are continuous on all of $\mathbb{R}$. $\square$

The next lemma gives sufficient conditions for the intersection of two continuous functions.

Lemma 144. *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions. Suppose there are real constants η, μ, λ, and ν such the following two conditions are satisfied.*

$$\varphi(x) \geq \eta \quad \textit{whenever} \quad x \leq \mu. \tag{5.7}$$

$$\varphi(x) \leq \lambda \quad \textit{whenever} \quad x \geq \nu. \tag{5.8}$$

Furthermore, suppose

$$\lim_{x\to\infty} g(x) = \infty \qquad \textit{and} \qquad \lim_{x\to-\infty} g(x) = -\infty. \tag{5.9}$$

Then there exists $x_0 \in \mathbb{R}$ *such that* $g(x_0) = \varphi(x_0)$*.*

Proof. By way of contradiction, suppose $g(x) \neq \varphi(x)$ for all $x \in \mathbb{R}$. We claim that one of the following inequalities must hold.

$$g(x) \leq \varphi(x) \qquad \text{for all} \qquad x \in \mathbb{R}. \tag{5.10}$$

$$g(x) \geq \varphi(x) \qquad \text{for all} \qquad x \in \mathbb{R}. \tag{5.11}$$

Indeed, if both (5.10) and (5.11) fail to hold, then there exists real numbers x_1 and x_2 such that $\varphi(x_1) - g(x_1)$ and $\varphi(x_2) - g(x_2)$ have opposite sign. But then, by the Intermediate Value Theorem, there exists $x_0 \in \mathbb{R}$ with $\varphi(x_0) = g(x_0)$, which contradicts our assumption.

If (5.10) holds then $\lim\limits_{x\to\infty} \varphi(x) = \infty$, contradicting (5.8). Similarly, if (5.11) holds then $\lim\limits_{x\to-\infty} \varphi(x) = -\infty$, contradicting (5.7). Therefore, there exists $x_0 \in \mathbb{R}$ such that $g(x_0) = \varphi(x_0)$. □

The generalization of Pólya's curve theorem will require the following technical results. In the next lemma, we adopt the convention that

$$c \cdot \infty = \begin{cases} \infty & \text{if } c \in (0, \infty] \\ -\infty & \text{if } c \in [-\infty, 0) \end{cases}$$

and

$$c \cdot (-\infty) = \begin{cases} -\infty & \text{if } c \in (0, \infty] \\ \infty & \text{if } c \in [-\infty, 0). \end{cases}$$

Lemma 145. *Suppose* $\omega \in [-\infty, \infty]$ *and* $c \in (-\infty, 0) \cup (0, \infty)$. *If* $\lim_{x\to\omega} \frac{\varphi(x)}{x} = c$, *then* $\lim_{x\to\omega} \varphi(x) = \omega \cdot c$

Proof. In the case where ω is finite, Lemma 145 follows from the product rule for limits. More specifically, for $\omega \in (-\infty, \infty)$,

$$\omega \cdot c = \left(\lim_{x\to\omega} x\right)\left(\lim_{x\to\omega} \frac{\varphi(x)}{x}\right) = \lim_{x\to\omega} x\frac{\varphi(x)}{x} = \lim_{x\to\omega} \varphi(x).$$

Now suppose $\omega = \infty$ and $c > 0$. Let $N > 0$ be given. It will be shown that there exists $M > 0$ such that

$$\varphi(x) > N \quad \text{whenever} \quad x > M.$$

Since

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = c,$$

there exists $M^* > 0$ such that

$$-\frac{c}{2} < \left(\frac{\varphi(x)}{x} - c\right) < \frac{c}{2} \quad \text{whenever} \quad x > M^*.$$

or, since $x > M^* > 0$,

$$\left(\frac{c}{2}\right) x < \varphi(x) < \left(\frac{3c}{2}\right) x \quad \text{whenever} \quad x > M^*.$$

Set $M = \max\left\{M^*, \frac{2N}{c}\right\}$. Then, for all $x > M$,

$$\varphi(x) > \left(\frac{c}{2}\right) x > \left(\frac{c}{2}\right)\left(\frac{2N}{c}\right) = N.$$

Thus, for every $N > 0$ there is a corresponding $M > 0$ such that $\varphi(x) > N$ whenever $x > M$. Whence

$$\lim_{x\to\infty} \varphi(x) = \infty = \infty \cdot c.$$

In the case where $\omega = \infty$ and $c < 0$ we have

$$\lim_{x\to\infty} \frac{-\varphi(x)}{x} = -c > 0.$$

Thus, by what was proved already, $\lim_{x\to\infty} -\varphi(x) = \infty \cdot (-c)$. Therefore

$$\lim_{x\to\infty} \varphi(x) = \infty \cdot c.$$

Finally, in the case where $\omega = -\infty$ and $c \in (-\infty, 0) \cup (0, \infty)$ we have that

$$\lim_{x\to\infty} \frac{-\varphi(-x)}{x} = \lim_{x\to\infty} \frac{\varphi(-x)}{-x} = \lim_{x\to-\infty} \frac{\varphi(x)}{x} = c.$$

Thus, by what was proved already,

$$\lim_{x\to\infty} -\varphi(-x) = \infty \cdot c.$$

Therefore

$$\lim_{x\to-\infty} \varphi(x) = \lim_{x\to\infty} \varphi(-x) = -\infty \cdot c.$$

$\square$

The following consequence of Schur's theorem (Theorem 39) will also be of use to us in the proof of the curve theorem.

Lemma 146. *Let m and n be positive integers and suppose that all the zeros of the polynomial*

$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \qquad (b_n \neq 0)$$

are real and negative. Then all the zeros of the polynomial

$$h(x) = \sum_{j=0}^{k} \frac{m!}{(m-j)!} b_j x^{m-j} \qquad \big(k = \min\{m, n\}\big)$$

are all real. Furthermore, if $m \leq n$, then all the zeros of $h(x)$ are all negative, and, if $m > n$, then $x = 0$ is a zero of $h(x)$ multiplicity $m - n$ and the remaining n zeros of $h(x)$ are all negative.

Proof. Without loss of generality, it may be assumed that each of the coefficients $b_0, b_1, b_2, \ldots, b_n$ are (strictly) positive. Since the zeros of

$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

are all real and the zeros of

$$(1 + x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j$$

are all real and of the same sign, it follows from Schur's theorem (Theorem 39) that the zeros of

$$f(x) = \sum_{j=0}^{k} j! \binom{m}{j} b_j x^j = \sum_{j=0}^{k} \frac{m!}{(m-j)!} b_j x^j \qquad (k = \min\{m, n\}) \tag{5.12}$$

are all real. Thus, all the zeros of

$$f^*(x) = \sum_{j=0}^{k} \frac{m!}{(m-j)!} b_j x^{k-j}$$

are real (see Lemma 43). Furthermore, since all the coefficients of $f^*(x)$ are positive, the zeros of $f^*(x)$ must, in fact, be negative. Finally, the remaining assertions of the corollary follow from the relation

$$h(x) = \begin{cases} f^*(x) & \text{if } m \leq n; \\ x^{m-n} f^*(x) & \text{if } m > n. \end{cases}$$

$\square$

5.3 Generalization of Pólya's Curve Theorem

The goal of this section is to prove the following theorem.

Theorem 147. *Fix $\alpha \geq 0$ and suppose $q(x) = \sum_{k=0}^{n} b_k x^k \in \mathcal{L}-\mathcal{P}^+$. Furthermore, suppose $f(x) = \sum_{k=0}^{m} a_k x^k$ ($a_m \neq 0$) is a real polynomial having only real zeros and form the polynomial*

$$g(x) = \left(\sum_{k=0}^{n} b_k (x - \alpha D)^k D^k \right) [f(x)], \tag{5.13}$$

where D denotes differentiation with respect to x. Then an explicit formula for $g(x)$ is

$$g(x) = \sum_{k=0}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} (-\alpha)^j \mathcal{H}_{k-j}^{(\alpha)}(x) f^{(k+j)}(x), \tag{5.14}$$

where $\mathcal{H}_n^{(\alpha)}(x)$ denotes the n^{th} generalized Hermite polynomial defined by the generating relation (2.11). Furthermore, if we define

$$F(x,y) = \sum_{k=0}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} (-\alpha)^j \mathcal{H}_{k-j}^{(\alpha)}(x) f^{(k+j)}(y), \tag{5.15}$$

then, for all $r, u \geq 0$ with $r + u > 0$ and all $s, v \in \mathbb{R}$, the polynomial $F(rx+s, ux+v)$ has only real zeros. In particular, $g(x) = F(x,x)$ has only real zeros.

Remark 148. Before giving the proof, we remark that, in the case where $\alpha = 0$, Theorem 147 essentially reduces to Pólya's curve theorem (Theorem 136), except we have assumed that all of the coefficients b_k are non-negative and we have not placed

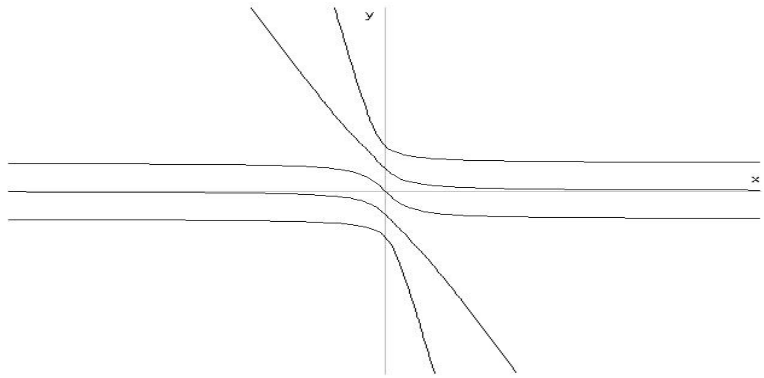

Figure 2: The curve $F(x,y)=0$ for $\deg(q)<\deg(f)$.

any restrictions on the degree of the polynomial $f(x)$. Furthermore, we have stated the conclusion in terms of zeros of a certain polynomial, rather than intersections of certain curves and lines. Figure 2 below demonstrates the curve $F(x,y)=0$ for $q(x)=(1+x)^2$ and $f(x)=\prod_{k=-2}^{2}(x-k)$. Note that there are $2=\deg(q)$ curves which diverge and $3=\deg(f''(x))$ curves which tend to finite limits (actually, to the zeros of $f''(y)$) as $x\to\pm\infty$.

Proof. We first note that the explicit formula (5.14) for $g(x)$ follows immediately from the definition (5.13) of $g(x)$ and Lemma 28.

The proof will be divided into four cases. To ease notation, the superscript (α) of $\mathcal{H}_n^{(\alpha)}(x)$ will be suppressed throughout the majority of the proof.

Suppose, first, that $\alpha>0$, $q(x)$ has only real negative zeros, and $f(x)$ has only real simple zeros. Let

$$p_x(z)=\sum_{k=0}^{n}b_k\sum_{j=0}^{k}\binom{k}{j}(-\alpha)^j\mathcal{H}_{k-j}(x)z^{k+j}. \tag{5.16}$$

By the addition formula (2.20) satisfied by the generalized Hermite polynomials, equation (5.16) becomes

$$\begin{aligned} p_x(z) &= \sum_{k=0}^{n} b_k z^k \sum_{j=0}^{k} \binom{k}{j} \mathcal{H}_{k-j}(x)(-\alpha z)^j \\ &= \sum_{k=0}^{n} b_k z^k \mathcal{H}_k(x-\alpha z). \end{aligned}$$

Fix $x_0 \in \mathbb{R}$. Since differentiation is translation invariant, the Rodrigues formula (2.12) yields

$$\mathcal{H}_k(x_0+z) = (-\alpha)^k \exp\left(\frac{(x_0+z)^2}{2\alpha}\right) \frac{d^k}{dz^k} \exp\left(-\frac{(x_0+z)^2}{2\alpha}\right)$$

holds for every nonnegative integer k. Since the zeros of $q(x) = \sum_{k=0}^{n} b_k x^k$ are all real and negative, the linear operator $\sum_{k=0}^{n} b_k z^k \frac{d^k}{dz^k}$ preserves reality of zeros (see Corollary 141) and, therefore, maps the entire Laguerre-Pólya class into itself (see Proposition 71). Since the function

$$\exp\left(-\frac{(x_0+z)^2}{2\alpha}\right)$$

belongs to the Laguerre-Pólya class,

$$\begin{aligned} \sum_{k=0}^{n} b_k z^k \frac{d^k}{dz^k} \exp\left(-\frac{(x_0+z)^2}{2\alpha}\right) &= \sum_{k=0}^{n} b_k \left(-\frac{z}{\alpha}\right)^k \exp\left(-\frac{(x_0+z)^2}{2\alpha}\right) \mathcal{H}_k(x_0+z) \\ &= \exp\left(-\frac{(x_0+z)^2}{2\alpha}\right) \sum_{k=0}^{n} b_k \left(-\frac{z}{\alpha}\right)^k \mathcal{H}_k(x_0+z) \end{aligned}$$

has only real zeros. Replacing z by $-\alpha z$, it follows that the function

$$\exp\left(-\frac{(x_0-\alpha z)^2}{2\alpha}\right) \sum_{k=0}^{n} b_k z^k \mathcal{H}_k(x_0-\alpha z)$$

has only real zeros. Therefore, all the zeros of the polynomial

$$\sum_{k=0}^{n} b_k z^k \mathcal{H}_k(x_0 - \alpha z) = p_{x_0}(z)$$

must also be real. Furthermore, the constant term of $p_x(z)$, as a polynomial in z, is equal to b_0, which is non-zero since we have assumed that the zeros of $q(x) = \sum_{k=0}^{n} b_k x^k$ are all negative. Therefore, the hypotheses of Proposition 143 are satisfied and so there are m continuous functions $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, such that

$$F(x, y) = b_0 a_m \prod_{k=1}^{m} (y - \varphi_k(x)) \tag{5.17}$$

and, for each $x \in \mathbb{R}$,

$$\varphi_1(x) < \varphi_2(x) < \cdots < \varphi_m(x).$$

It will now be shown that each line of positive slope must intersect each of the functions $\varphi_k(x)$, $k = 1, 2, 3, \ldots, m$. By Proposition 144, it suffices to show that there are constants $\eta, \lambda, \mu, \nu \in \mathbb{R}$ such that

$$\varphi_k(x) > \eta \quad \text{whenever} \quad x < \mu \qquad (k = 1, 2, 3, \ldots, m) \tag{5.18}$$

and

$$\varphi_k(x) < \lambda \quad \text{whenever} \quad x > \nu \qquad (k = 1, 2, 3, \ldots, m). \tag{5.19}$$

A detailed analysis of the asymptotic behavior of the functions $\varphi_k(x)$ will be achieved by examining the function $F(x, xt)$, which by equation (5.17), has the representation

$$F(x, xt) = b_0 a_m \prod_{k=1}^{m} (xt - \varphi_k(x)).$$

On the other hand, rewriting $F(x, y)$, as defined in equation (5.15), without the use of summation notation yields

$$
\begin{aligned}
F(x,y) = {} & b_0 \left[\mathcal{H}_0(x) f(y)\right] \\
& + b_1 \left[\mathcal{H}_1(x) f'(y) - \alpha \mathcal{H}_0(x) f''(y)\right] \\
& + b_2 \left[\mathcal{H}_2(x) f''(y) - 2\alpha \mathcal{H}_1(x) f'''(y) + \mathcal{H}_0(x) f^{(4)}(y)\right] \\
& . \\
& . \\
& . \\
& + b_n \left[\mathcal{H}_n(x) f^{(n)}(y) - \alpha n \mathcal{H}_{n-1}(x) f^{(n+1)}(y) + \cdots + (-\alpha)^n \mathcal{H}_0(x) f^{(2n)}(y)\right].
\end{aligned}
$$

Note that, for all $k \in \{0, 1, 2, \dots\}$, the leading term of $\mathcal{H}_k(x)$ is x^k. Thus another representation of $F(x, xt)$ is given by

$$F(x, xt) = a_m x^m \left(\sum_{k=0}^{\min\{m,n\}} \frac{m!}{(m-k)!} b_k t^{m-k} \right) + h_1(x,t),$$

where $h_1(x,t)$ is a polynomial in x and t (and α) which has degree, as a polynomial in x, which is strictly less than m. By Proposition 142, the zeros of

$$\frac{F(x,xt)}{x^m} = b_0 a_m \prod_{k=1}^{m} \left(t - \frac{\varphi_k(x)}{x} \right) = a_m \left(\sum_{k=0}^{\min\{m,n\}} \frac{m!}{(m-k)!} b_k t^{m-k} \right) + \frac{h_1(x,t)}{x^m}$$

converge to the zeros of

$$a_m \left(\sum_{k=0}^{\min\{m,n\}} \frac{m!}{(m-k)!} b_k t^{m-k} \right) \tag{5.20}$$

as $x \to \infty$.

If $m \leq n$ then, by Lemma 146, the zeros of (5.20) are all negative. Thus, there are negative constants $t_1, t_2, \dots, t_m$ such that

$$\lim_{x \to \pm\infty} \frac{\varphi_k(x)}{x} = t_k \qquad (k = 1, 2, 3, \dots, m).$$

Therefore, by Lemma 145,

$$\lim_{x\to\infty}\varphi_k(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty}\varphi_k(x) = \infty \qquad (k = 1, 2, 3, \ldots, m).$$

Whence, if we set $\eta = 0$ and $\lambda = 0$, then there are real constants μ and ν such that

$$\varphi_k(x) > \eta \quad \text{whenever} \quad x < \mu \qquad (k = 1, 2, 3, \ldots, m),$$

and

$$\varphi_k(x) < \lambda \quad \text{whenever} \quad x > \nu \qquad (k = 1, 2, 3, \ldots, m),$$

Therefore, (5.18) and (5.19) are satisfied and, by Lemma 144, each line of positive slope must intersect each of the functions $\varphi_k(x)$, $k = 1, 2, 3, \ldots, m$, at least once.

If $m > n$, then, by Lemma 146, the polynomial (5.20) has n negative zeros, which we denote by $s_1 \leq s_2 \leq \cdots s_n < 0$, and a zero of multiplicity $m - n$ at the origin. Since

$$\frac{\varphi_1(x)}{x} < \frac{\varphi_2(x)}{x} < \cdots < \frac{\varphi_n(x)}{x} < \cdots < \frac{\varphi_m(x)}{x} \quad \text{whenever} \quad x > 0,$$

it follows that

$$\lim_{x\to\infty}\frac{\varphi(x)}{x} = \begin{cases} s_k & k = 1,\ 2,\ 3,\ \ldots,\ n \\ 0 & k = n+1,\ n+2,\ n+3,\ \ldots,\ m. \end{cases}$$

By Lemma 145,

$$\lim_{x\to\infty}\varphi_k(x) = -\infty \qquad (k = 1, 2, 3, \ldots, n),$$

and the asymptotic behavior of the remaining $m - n$ zeros remains in question. Similarly, since

$$\frac{\varphi_1(x)}{x} > \frac{\varphi_2(x)}{x} > \cdots > \frac{\varphi_{m-n+1}(x)}{x} > \cdots > \frac{\varphi_m(x)}{x} \quad \text{whenever} \quad x < 0,$$

it follows that

$$\lim_{x\to-\infty}\frac{\varphi(x)}{x}=\begin{cases}0 & k=1,\,2,\,3,\,\ldots,\,m-n\\ s_{m+1-k} & k=m-n+1,\,m-n+2,\,m-n+3,\,\ldots,\,m.\end{cases}$$

Again, by Lemma 145,

$$\lim_{x\to-\infty}\varphi_k(x)=\infty \qquad (k=m-n+1,m-n+2,\ldots,m),$$

and the asymptotic behavior of the remaining $m-n$ zeros remains in question.

Note that, by examining the definition of $F(x,y)$ given in equation (5.15),

$$F(x,y)=b_nf^{(n)}(y)x^n+h_2(x,y)$$

where $h_2(x,y)$ is a polynomial in x and y (and α) which has degree, as a polynomial in x, which is less than m. Thus, as a polynomial in y, the Taylor coefficients of

$$\frac{F(x,y)}{x^m}$$

tend to those of $b_nf^{(n)}(y)$ as $x\to\pm\infty$. Since $f(y)$ (of degree m) has only real simple zeros, $f^{(n)}(y)$ also has only simple real zeros (and there are exactly $m-n$ of them). By Proposition 142, for any open interval containing a zero of $f^{(n)}(y)$, there is a zero of $F(x,y)/x^m$, and therefore a zero of $F(x,y)$, for all sufficiently large x. Thus, the $m-n$ functions $\varphi_k(x)$ whose asymptotic behavior remained in question must, in fact, tend to the zeros of $f^{(n)}(y)$ as $x\to\pm\infty$.

To summarize, denote the zeros of $f^{(n)}(y)$ by $z_1<z_2<\cdots<z_{m-n}$. Then,

$$\lim_{x\to\infty}\varphi(x)=\begin{cases}-\infty & k=1,\,2,\,3,\,\ldots,\,n\\ z_{k-n} & k=n+1,\,n+2,\,n+3,\,\ldots,\,m\end{cases}$$

and

$$\lim_{x\to-\infty}\varphi(x)=\begin{cases}z_k & k=1,\,2,\,3,\,\ldots,\,m-n\\ \infty & k=m-n+1,\,m-n+2,\,m-n+3,\,\ldots,\,m.\end{cases}$$

Thus, each of the functions $\varphi_k(x)$ can be made to be greater than

$$\eta = \left(\min \left\{ y : f^{(m)}(y) = 0 \right\} - 1 \right)$$

by taking x to be sufficiently large and negative. Similarly, each of the functions $\varphi_k(x)$ can be made to be less than

$$\lambda = \left(\max \left\{ y : f^{(m)}(y) = 0 \right\} + 1 \right)$$

by taking x to be sufficiently large. Thus, equations (5.18) and (5.19) are satisfied and, therefore, just as in the case when $m \leq n$, each line of positive slope intersects each of the functions $\varphi_k(x)$, $k = 1, 2, 3, \ldots, m$, at least once.

The theorem will now be completed for this special case where $\alpha > 0$, $q(x)$ has only real negative zeros, and $f(x)$ has only simple real zeros. Let r and u be positive real constants and let s and t be arbitrary real constants. Then the line

$$y = \frac{u}{r}(x - s) + v$$

intersects each of the functions $\varphi_k(x)$, $k = 1, 2, 3, \ldots, m$, at least once. Each intersection corresponds to a zero of the polynomial

$$F\left(x, \frac{u}{r}(x - s) + v\right), \tag{5.21}$$

which is of degree m. Therefore, the polynomial (5.21) has only real zeros. The transformation $x \to (rx + s)$ will not introduce any non-real zeros and, therefore, $F(rx + s, ux + v)$ has only real zeros. If one of r or u is zero, then we consider the sequence of polynomials

$$F\left(\frac{1}{n}x + s, ux + v\right) \quad \text{or} \quad F\left(rx + s, \frac{1}{n}x + v\right),$$

respectively, which converge uniformly on compact subsets of $\mathbb{C}$ to $F(s, ux+v)$ or $F(rx+s, v)$, respectively. In either case, by Hurwitz' theorem, the zeros of $F(rx+s, ux+v)$ in this case are also all real.

Now consider the case where $\alpha > 0$, $q(x)$ has only real negative zeros, and $f(x)$ is an arbitrary polynomial having only real zeros. Write

$$f(x) = c_m \prod_{k=1}^{m} (x - x_k),$$

where $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_m$ are the zeros of $f(x)$. Then each of the polynomials

$$f_N(x) = c_m \prod_{k=1}^{m} \left(x - x_k - \frac{k}{N} \right) \qquad (N = 1, 2, 3, \dots)$$

has only simple real zeros and $f_N(x) \to f(x)$ uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$. For each $N = 1, 2, 3, \dots,$ let

$$F_N(x, y) = \sum_{k=0}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} (-\alpha)^j \mathcal{H}_{k-j}(x) f_N^{(k+j)}(y). \tag{5.22}$$

If $r, u \geq 0$ with $r + u > 0$ and $s, v \in \mathbb{R}$, then, by the previous case, the polynomial $F_N(rx+s, ux+v)$ has only real zeros. Since $F_N(rx+s, ux+v) \to F(rx+s, ux+v)$ uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$, by Hurwitz' theorem, $F(rx+s, ux+v)$ has only real zeros.

Now consider the case where $\alpha > 0$, $q(x)$ has only real *non-positive* zeros, and $f(x)$ is a polynomial having only real zeros. Then each of the polynomials

$$q_N(x) = q\left(x + \frac{1}{N} \right) = \sum_{k=0}^{n} b_{k,N} x^k \qquad (N = 1, 2, 3, \dots)$$

has only real negative zeros. Set

$$F_N(x, y) = \sum_{k=0}^{n} b_{k,N} \sum_{j=0}^{k} \binom{k}{j} (-\alpha)^j \mathcal{H}_{k-j}(x) f^{(k+j)}(y).$$

If $r, u \geq 0$ with $r + u > 0$ and $s, v \in \mathbb{R}$, then, by the previous case, the polynomial $F_N(rx + s, ux + v)$ has only real zeros. Since $F_N(rx + s, ux + v) \to F(rx + s, ux + v)$ uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$, by Hurwitz' theorem, $F(rx+s, ux+v)$ also has only real zeros.

Finally, in the case where $\alpha = 0$ (for which we will need to re-introduce the superscript on the generalized Hermite polynomials), $q(x)$ has only real non-positive zeros, and $f(x)$ is a polynomial having only real zeros. Set

$$F_N(x, y) = \sum_{k=0}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} \left(-\frac{1}{N}\right)^j \mathcal{H}_{k-j}^{(1/N)}(x) f^{(k+j)}(y).$$

If $r, u \geq 0$ with $r + u > 0$ and $s, v \in \mathbb{R}$, then, by the previous case, the polynomial $F_N(rx+s, ux+v)$ has only real zeros. Noting that $\mathcal{H}_n^{(1/N)} \to x^n$ as $N \to \infty$, it follows that $F_N(rx + s, ux + v) \to F(rx + s, ux + v)$ uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$. Thus, by Hurwitz' theorem, $F(rx + s, ux + v)$ has only real zeros.

□

5.4 Classification of Hermite Multiplier Sequences

In the previous chapter, we saw that every H-multiplier sequence is also a multiplier sequence (Proposition 118). Furthermore, we saw that every non-trivial H-multiplier sequence whose elements are all non-negative must be non-decreasing (Theorem 127). Conversely, we conjectured, but did not prove, that every non-negative multiplier sequence which is non-decreasing must also be an H-multiplier sequence (Remark 134).

We are now in a position to prove this conjecture.

Proposition 149. *Suppose*

$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots b_n x^n \in \mathcal{L} - \mathcal{P}^+$$

and let

$$\widetilde{q}(x) = \sum_{k=0}^{n} b_k k! \binom{x}{k} = b_0 + b_1 x + b_2 x(x-1) + \cdots + b_n \prod_{k=1}^{n} (x - k + 1).$$

Then the sequence $\{\widetilde{q}(k)\}_{k=0}^{\infty}$ *is an* H*-multiplier sequence.*

Proof. Let T_H denote the T_H-operator associated with the sequence $\{\widetilde{q}(k)\}_{k=0}^{\infty}$. We claim that the operator T_H can be represented as

$$T_H = \sum_{k=0}^{n} b_k \left(x - \frac{1}{2}D\right)^k D^k.$$

To see this, we set

$$\delta = xD - \frac{1}{2}D^2,$$

and, from Hermite's Differential equation (2.7), we have

$$\delta[H_n(x)] = nH_n(x) \qquad (n = 0, 1, 2, \dots).$$

By Lemma 26, for any integers $n \geq 0$ and $k \geq 1$,

$$\begin{aligned} \left(\left(x - \frac{1}{2}D\right)^k D^k\right)[H_n(x)] &= \big(\delta(\delta-1)(\delta-2)\cdots(\delta-k+1)\big)[H_n(x)] \\ &= n(n-1)(n-2)\cdots(n-k+1)H_n(x). \end{aligned}$$

Therefore, for any non-negative integers n and k,

$$\left(\left(x - \frac{1}{2}D\right)^k D^k\right)[H_n(x)] = k!\binom{n}{k} H_n(x),$$

from which we obtain

$$\left(\sum_{k=0}^{n} b_k \left(x-\frac{1}{2}D\right)^k D^k\right)[H_n(x)] = \sum_{k=0}^{n} b_k k! \binom{n}{k} H_n(x) = \widetilde{q}(n) H_n(x).$$

Let $f(x)$ be a real polynomial with only real zeros. Then, by Theorem 147, the polynomial

$$g(x) = \left(\sum_{k=0}^{n} b_k \left(x-\frac{1}{2}D\right)^k D^k\right)[f(x)] = T_H[f(x)]$$

has only real zeros. $\square$

Remark 150. It can now be shown that, just as in the classical setting, there are H-multiplier sequences which are *not* H-CZDS. Indeed, if $q(x) = x^2 + 2x + 1$, then $\widetilde{q}(x) = 1 + x + x^2$, whence the sequence $\left\{1 + k + k^2\right\}_{k=0}^{\infty}$ is an H-multiplier sequence. However, the sequence is not a (classical) CZDS and so, by Proposition 109, it cannot be an H-CZDS. Alternatively, the polynomial

$$f(x) = (x-12)^9(x^2+12)$$

has two non-real zeros, while it can be shown that $T_H[f(x)]$, which is equal to

$$(x-12)^5(133x^6 - 4008x^5 + 39681x^4 - 147744x^3 + 240876x^2 - 162432x + 40176),$$

has four non-real zeros.

It will now be shown that every non-negative multiplier sequence which is non-decreasing must be an H-multiplier sequence.

Proposition 151. *If the sequence of non-negative real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is a non-decreasing multiplier sequence, then $\{\gamma_k\}_{k=0}^{\infty}$ is also an H-multiplier sequence.*

Proof. By the transcendental characterization of multiplier sequences (Theorem 46), the function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}-\mathcal{P}^+$. Thus (see Remark 21) $\varphi(x)$ has a product representation of the form

$$\varphi(x) = cx^s e^{\sigma x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right), \tag{5.23}$$

where $c \in \mathbb{R}$, s is a non-negative integer, $\sigma \geq 0$, $x_k > 0$, $0 \leq \omega \leq \infty$, and $\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$. Furthermore, by assumption, the inequality $0 \leq \gamma_k \leq \gamma_{k+1}$ holds for every integer $k \geq 0$. Thus, by Lemma 23, we actually have $\sigma \geq 1$ in equation (5.23).

Let us first consider the case when $\sigma = 1$ and $\omega < \infty$. In this case, set

$$q_1(x) = cx^s \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right).$$

Then, by Lemma 138, $\gamma_k = \widetilde{q_1}(k)$ $(k = 0, 1, 2, \dots)$. Therefore, by Proposition 149, $\{\gamma_k\}_{k=0}^{\infty}$ is an H-multiplier sequence.

We will now consider the case when $\sigma > 1$ and $\omega < \infty$. In this case we have

$$\varphi\left(\frac{x}{\sigma}\right) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} \left(\frac{x}{\sigma}\right)^k = c\left(\frac{x}{\sigma}\right)^s e^x \prod_{k=1}^{\omega} \left(1 + \frac{x}{\sigma x_k}\right).$$

Define the polynomial $q_2(x)$ by

$$q_2(x) = c\left(\frac{x}{\sigma}\right)^s \prod_{k=1}^{\omega} \left(1 + \frac{x}{\sigma x_k}\right).$$

Then, by Lemma 138,

$$\frac{\gamma_k}{\sigma^k} = \widetilde{q_2}(k) \qquad (k = 0, 1, 2, \dots),$$

whence $\gamma_k = \sigma^k \widetilde{q_2}(k)$ $(k = 0, 1, 2, \dots)$. Since the polynomial $q_2(x) \in \mathcal{L}-\mathcal{P}^+$, the sequence $\{\widetilde{q}(k)\}_{k=0}^{\infty}$ is an H-multiplier sequence. Since $\sigma > 1$, the sequence $\left\{\sigma^k\right\}_{k=0}^{\infty}$ is also an H-multiplier sequence (see Corollary 129). Therefore the Hadamard product of these two sequences $\left\{\sigma^k\, \widetilde{q_2}(k)\right\}_{k=0}^{\infty} = \{\gamma_k\}_{k=0}^{\infty}$ is also an H-multiplier sequence.

Finally, we consider the case when $\sigma \geq 1$ and $\omega = \infty$. In this case, consider the sequences $\{\gamma_{m,k}\}_{k=0}^{\infty}$ defined by

$$\varphi_m(x) = cx^s e^{\sigma x} \prod_{k=1}^{m} \left(1 + \frac{x}{x_k}\right) = \sum_{k=0}^{\infty} \frac{\gamma_{m,k}}{k!} x^k \qquad (m = 1, 2, 3, \dots).$$

Then, by the previous cases, for each m, the sequence $\{\gamma_{m,k}\}_{k=0}^{\infty}$ is an Hermite multiplier sequence. Since, for every integer $k \geq 0$, $\gamma_{m,k} \to \gamma_k$ as $m \to \infty$, it follows from Hurwitz' theorem, that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is also an Hermite multiplier sequence. $\square$

Let us combine the preceding results to give a complete characterization of H-multiplier sequences.

Theorem 152. (Characterization of non-trivial H-multiplier sequences) *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. Then the following are equivalent.*

(1) *$\{\gamma_k\}_{k=0}^{\infty}$ is a non-decreasing multiplier sequence.*

(2) *$\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial H-multiplier sequence.*

(3) *The function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is an entire function which has a product representation of the form*

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right),$$

where $c \in \mathbb{R}$*,* m *is a non-negative integer,* $\sigma \geq 1$*,* $x_k > 0$*,* $0 \leq \omega \leq \infty$*, and* $\displaystyle\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$*.*

Proof. The equivalence of (1) and (3) is an immediate consequence of the transcendental characterization of multiplier sequences (Theorem 46) and the special form which the product representation of a function in $\mathcal{L}-\mathcal{P}^{+}$ must have when its Taylor coefficients are non-decreasing (Lemma 23).

The equivalence of (1) and (2) is an immediate consequence of the fact that every H-multiplier sequence is a multiplier sequence (Proposition 118), every non-trivial non-negative H-multiplier sequence is non-decreasing (Proposition 127), and the fact that every non-negative multiplier sequence which is non-decreasing is an H-multiplier sequence (Proposition 151). $\square$

Remark 153. If $\{\gamma_k\}_{k=0}^{\infty}$ is a non-trivial H-multiplier sequence, then, by Proposition 119, the sequences $\{-\gamma_k\}_{k=0}^{\infty}$, $\left\{(-1)^k\gamma_k\right\}_{k=0}^{\infty}$, and $\left\{(-1)^{k+1}\gamma_k\right\}_{k=0}^{\infty}$ are also H-multiplier sequences, and one of these sequences consists entirely of non-negative elements. Therefore, Theorem 152 is a *complete* characterization of non-trivial H-multiplier sequences, as claimed.

Chapter 6
Linear Operators for Other Polynomial Sets

6.1 General Polynomial Sets

In this chapter we consider the analogous results of the preceding chapters for other polynomial sets. We begin in a very general setting and will then apply these results to various polynomial sets. We begin with the relevant definitions.

Definition 154. A set of polynomials $\{q_k(x)\}_{k=0}^{\infty}$ satisfying

$$\deg(q_k(x)) = k \qquad (k = 0, 1, 2, \dots).$$

is called a *simple set* of polynomials.

Since we adopt the convention that $\deg(0) < 0$, any simple set of real polynomials forms a basis for the vector space of real polynomials $\mathbb{R}[x]$.

Definition 155. Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of real polynomials and let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then the T_Q*-operator associated with the sequence* $\{\gamma_k\}_{k=0}^{\infty}$ is the linear operator defined on $\mathbb{R}[x]$ by

$$T_Q[q_n(x)] = nq_n(x) \qquad (n = 0, 1, 2, \dots).$$

If the operator T_Q has the property that

$$T_Q[p(x)] \in \mathcal{L} - \mathcal{P} \qquad \text{whenever} \qquad p(x) \in \mathbb{R}[x] \cap \mathcal{L} - \mathcal{P},$$

then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier sequence for the simple set* Q or, for brevity, a Q-*multiplier sequence*. Similarly, if the operator T_Q has the property that, for any real polynomial $p(x)$,

$$Z_C\big(T_Q[p(x)]\big) \leq Z_C\big(p(x)\big),$$

(where $Z_C(f(x))$ is defined as in Notation 56) then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *complex zero decreasing sequence for the simple set* Q or, for brevity, a Q-*CZDS*.

The next proposition shows that the complex zero decreasing sequences for a given simple set are the same as those for another related simple set.

Lemma 156. *Let* $Q = \{q_k(x)\}_{k=0}^{\infty}$ *be a simple set of polynomials and suppose* $\{c_k\}_{k=0}^{\infty}$ *is a sequence of non-zero real numbers,* $\alpha \in \mathbb{R} \setminus \{0\}$*, and* $\beta \in \mathbb{R}$*. Let* $\widehat{Q} = \{\widehat{q}(x)\}_{k=0}^{\infty}$*, where we define*

$$\widehat{q}_k(x) = c_k\, q_k(\alpha x + \beta) \qquad (k = 0, 1, 2, \dots).$$

Then $\{\gamma_k\}_{k=0}^{\infty}$ *is a* Q*-CZDS if and only if* $\{\gamma_k\}_{k=0}^{\infty}$ *is a* $\widehat{Q}$*-CZDS.*

Proof. Since we have assumed that Q is a simple set of polynomials, $\{c_k\}_{k=0}^{\infty}$ is a sequence of non-zero real numbers, and $\alpha \in \mathbb{R}$ is non-zero, it follows that $\widehat{Q}$ is also a simple set of polynomials.

We will make frequent use of the transformations

$$x \longmapsto \alpha x + \beta \qquad \text{and} \qquad x \longmapsto \frac{x - \beta}{\alpha},$$

which , since α and β are real and $\alpha \neq 0$, do not change the number of non-real zeros of any polynomial.

Suppose $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-CZDS and let $\displaystyle\sum_{k=0}^{n} a_k \widehat{q}_k(x)$ be any real polynomial. Then, by the definition of $\widehat{q}_k(x)$ and by applying the transformation $x \mapsto \dfrac{x-\beta}{\alpha}$,

$$Z_C\left(\sum_{k=0}^{n} a_k \widehat{q}_k(x)\right) = Z_C\left(\sum_{k=0}^{n} a_k c_k q_k(\alpha x+\beta)\right) = Z_C\left(\sum_{k=0}^{n} a_k c_k q_k(x)\right). \tag{6.1}$$

Since we have assumed $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-CZDS, we have

$$Z_C\left(\sum_{k=0}^{n} a_k c_k q_k(x)\right) \geq Z_C\left(\sum_{k=0}^{n} a_k c_k \gamma_k q_k(x)\right). \tag{6.2}$$

Applying the transformation $x \mapsto \alpha x+\beta$ yields

$$Z_C\left(\sum_{k=0}^{n} a_k c_k \gamma_k q_k(x)\right) = Z_C\left(\sum_{k=0}^{n} a_k \gamma_k c_k q_k(\alpha x+\beta)\right) = Z_C\left(\sum_{k=0}^{n} a_k \gamma_k \widehat{q}_k(x)\right). \tag{6.3}$$

Together, (6.1), (6.2), and (6.3) show that $\{\gamma_k\}_{k=0}^{\infty}$ is a $\widehat{Q}$-CZDS.

Conversely, suppose $\{\gamma_k\}_{k=0}^{\infty}$ is a $\widehat{Q}$-CZDS and let $\displaystyle\sum_{k=0}^{m} b_k q_k(x)$ be any real polynomial. Then, using the fact that $c_k \neq 0$ and applying the transformation $x \mapsto \alpha x+\beta$, we have

$$Z_C\left(\sum_{k=0}^{m} b_k q_k(x)\right) = Z_C\left(\sum_{k=0}^{m} \frac{b_k}{c_k} c_k q_k(\alpha x+\beta)\right) = Z_C\left(\sum_{k=0}^{m} \frac{b_k}{c_k} \widehat{q}_k(x)\right). \tag{6.4}$$

Since we have assumed $\{\gamma_k\}_{k=0}^{\infty}$ is a $\widehat{Q}$-CZDS, we have

$$Z_C\left(\sum_{k=0}^{m} \frac{b_k}{c_k} \widehat{q}_k(x)\right) \geq Z_C\left(\sum_{k=0}^{m} \frac{b_k}{c_k} \gamma_k \widehat{q}_k(x)\right) = Z_C\left(\sum_{k=0}^{m} \frac{b_k}{c_k} \gamma_k c_k q_k(\alpha x+\beta)\right). \tag{6.5}$$

Applying the transformation $x \mapsto \dfrac{x-\beta}{\alpha}$ yields

$$Z_C\left(\sum_{k=0}^{m} \frac{b_k}{c_k} \gamma_k c_k q_k(\alpha x+\beta)\right) = Z_C\left(\sum_{k=0}^{m} b_k \gamma_k q_k(x)\right). \tag{6.6}$$

Together, (6.4), (6.5), and (6.6) show that $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-CZDS. $\square$

As the next lemma shows, an analogous result holds for multiplier sequences for simple sets of polynomials.

Lemma 157. *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials and suppose $\{c_k\}_{k=0}^{\infty}$ is a sequence of non-zero real numbers, $\alpha \in \mathbb{R} \setminus \{0\}$, and $\beta \in \mathbb{R}$. Let $\widehat{Q} = \{\widehat{q}(x)\}_{k=0}^{\infty}$, where we define*

$$\widehat{q}_k(x) = c_k\, q_k(\alpha x + \beta) \qquad (k = 0, 1, 2, \dots).$$

Then $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-multiplier sequence if and only if $\{\gamma_k\}_{k=0}^{\infty}$ is a $\widehat{Q}$-multiplier sequence.

Proof. The proof is similar to that of Lemma 156, except we initially chose the polynomials $\sum_{k=0}^{n} a_k \widehat{q}_k(x)$ and $\sum_{k=0}^{m} b_k q_k(x)$ so that they have only real zeros. □

Remarkably, the preceding results allow us to conclude that every Q-multiplier sequence is also a (classical) multiplier sequence, regardless of the choice of the simple set Q.

Theorem 158. *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials. If the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-multiplier sequence, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a (classical) multiplier sequence.*

Proof. Choose the real constants $a_{k,j}$ such that

$$q_k(x) = \sum_{j=0}^{k} a_{k,j} x^j \qquad (k = 0, 1, 2, \dots).$$

Since the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence for the simple set $\{q_k(x)\}_{k=0}^{\infty}$, it is, by Lemma 157, also a multiplier sequence for each of the simple sets

$$P^{(\alpha)} = \left\{p_k^{(\alpha)}(x)\right\}_{k=0}^{\infty} \qquad (\alpha \in \mathbb{R} \setminus \{0\}),$$

which we define by

$$\left\{p_k^{(\alpha)}(x)\right\}_{k=0}^{\infty} = \left\{\frac{1}{\alpha^k a_{k,k}} q_k(\alpha x)\right\}_{k=0}^{\infty} = \left\{x^k + \sum_{j=0}^{k-1} \frac{a_{k,j}}{a_{k,k}} \frac{x^j}{\alpha^{k-j}}\right\}_{k=0}^{\infty}.$$

Suppose the real polynomial $f(x) = \sum_{k=0}^{n} b_k x^k$ has only real zeros. For each non-zero $\alpha \in \mathbb{R}$ we can expand $f(x)$ in terms of the basis $P^{(\alpha)}$ as

$$f(x) = \sum_{k=0}^{n} c_{\alpha,k} p_k^{(\alpha)}(x).$$

Since

$$\lim_{\alpha\to\infty} p_k^{(\alpha)}(x) = x^k \qquad (k = 0, 1, 2, \dots),$$

it follows that

$$\lim_{\alpha\to\infty} c_{\alpha,k} = b_k \qquad (k = 0, 1, 2, \dots, n).$$

Since $\{\gamma_k\}_{k=0}^{\infty}$ is a $P^{(\alpha)}$-multiplier sequence for each $\alpha \neq 0$, each of the functions

$$f_\alpha(x) = \sum_{k=0}^{n} c_{\alpha,k} \gamma_k p_k^{(\alpha)}(x) \qquad (\alpha = 1, 2, 3, \dots)$$

have only real zeros. Since the sequence of functions $\{f_\alpha(x)\}_{\alpha=1}^{\infty}$ converges uniformly on compact subsets of $\mathbb{C}$ to the polynomial $\sum_{k=0}^{n} b_k \gamma_k x^k$, we have, by Hurwitz' theorem, that $\sum_{k=0}^{n} b_k \gamma_k x^k$ has only real zeros. Therefore, the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a (classical) multiplier sequence. $\square$

As the next theorem shows, an analogous result holds for H-CZDS.

Theorem 159. *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of polynomials. If the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a Q-CZDS, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a (classical) CZDS.*

Proof. The proof follows along the same lines as the proof of Theorem 158, except we note that each of the functions $f_\alpha(x)$ satisfy $Z_c(f_\alpha) \leq Z_c(f)$. $\square$

Remark 160. Several remarks are in order.

1. The converse of Theorems 158 and 159 are false. For example, if we take $Q = H$ to be the set of Hermite polynomials, then, for $0 < r < 1$, the sequence $\left\{r^k\right\}_{k=0}^{\infty}$ is a CZDS and a multiplier sequence, but, since it is positive and decreasing, is *not* an H-CZDS or an H-multiplier sequence (see Theorem 127 and Remark 128).

2. Since the Hermite polynomials form a simple set of polynomials, we now have another proof that every H-CZDS is a classical CZDS (Proposition 109). Similarly, we have another proof that every H-multiplier sequence is a classical multiplier sequence (Proposition 118).

3. When studying multiplier sequences (or CZDS) for any simple set Q, we may restrict our attention to the classical multiplier sequences (or CZDS). Furthermore, several properties of multiplier sequences will carry over to Q-multiplier sequences, regardless of the choice of Q. For example, the elements of a Q-multiplier sequence must all have the same sign, or they must alternate signs.

4. The preceding results provide another tool for attempting to characterize the

classical CZDS. For example, we mentioned earlier that it is not known whether the sequence $\left\{e^{-k^3}\right\}_{k=0}^{\infty}$ is a CZDS or not. If we could show that there is a simple set Q such that $\left\{e^{-k^3}\right\}_{k=0}^{\infty}$ is a Q-CZDS, then it will follow that $\left\{e^{-k^3}\right\}_{k=0}^{\infty}$ is a classical CZDS.

6.2 Generalized Hermite Polynomials

In this section, we examine linear operators defined on the simple set of generalized Hermite polynomials $\mathcal{H}^{\alpha} = \left\{\mathcal{H}_k^{(\alpha)}(x)\right\}_{k=0}^{\infty}$ defined by the generating relation (2.11). Since the generalized Hermite polynomials satisfy the differential equation (2.17), we have the relation

$$\left(xD - \alpha D^2\right)[\mathcal{H}_n^{(\alpha)}(x)] = n\mathcal{H}_n^{(\alpha)}(x) \qquad (\alpha \in \mathbb{R};\ n = 0, 1, 2, \dots). \tag{6.7}$$

Therefore, for $\alpha > 0$, many of the results we have obtained for the Hermite polynomials which were a consequence of the relation

$$\left(xD - \frac{1}{2}D^2\right)[H_n(x)] = nH_n(x) \qquad (\alpha \in \mathbb{R};\ n = 0, 1, 2, \dots)$$

carry over to these generalized Hermite polynomials. For example, the same argument used to show that sequences of the form

$$\left\{\varphi(k)\left(k(k-1)\cdots(k-m+1)\prod_{i=1}^{p}(k-b_i)\right)\right\}_{k=0}^{\infty} \qquad (b_i \le m;\ i = 1, 2, 3, \dots, p)$$

are H-CZDS can be used to show that these sequences are also $\mathcal{H}^{\alpha}$-CZDS, where $\alpha > 0$ has been fixed. Similarly, the proof of the fact that every non-negative multiplier sequence which is non-decreasing must be an H-multiplier sequence can be adapted to the setting of $\mathcal{H}^\alpha$-multiplier sequences, where again $\alpha > 0$ has been fixed. Instead of reproducing these arguments, we present a more elegant proof which appeals to the results of the previous section.

Lemma 161. *Suppose $\alpha > 0$. Then a sequence of real numbers $\{\gamma_k\}_{k=0}^\infty$ is an H-multiplier sequence if and only if it is an $\mathcal{H}^\alpha$-multiplier sequence. Similarly, a sequence of real numbers $\{\gamma_k\}_{k=0}^\infty$ is an H-CZDS if and only if it is an $\mathcal{H}^\alpha$-CZDS.*

Proof. Let us recall the relation between the classical and generalized Hermite polynomials (equation (2.13)),

$$\mathcal{H}_k^{(\alpha)}(x) = \left(\frac{\alpha}{2}\right)^{k/2} H_k\left(\frac{x}{\sqrt{2\alpha}}\right) \qquad \big(\alpha \in (\mathbb{R} \setminus \{0\})\,;\, k = 0, 1, 2, \ldots\big).$$

From this relation and the fact that we have assumed $\alpha > 0$, we see that this lemma follows immediately from Lemmas 156 and 157. $\square$

By Lemma 161, all of the results of the preceding chapters apply to the simple set $\mathcal{H}^\alpha$ for any $\alpha > 0$. When $\alpha = 0$, the set $\mathcal{H}^{(0)}$ reduces to the standard basis and the classical results apply. Let us now investigate the case where $\alpha < 0$.

Example 162. The sequence $\{k\}_{k=0}^\infty$ is *not* an $\mathcal{H}^\alpha$-multiplier sequence for any $\alpha < 0$. Indeed, the polynomial

$$\mathcal{H}_2^{(\alpha)}(x) + \alpha\mathcal{H}_0^{(\alpha)}(x) = (x^2 - \alpha) + \alpha(1) = x^2$$

has only real zeros, while

$$2\mathcal{H}_2^{(\alpha)}(x) + \alpha\, 0\, \mathcal{H}_0^{(\alpha)}(x) = 2\mathcal{H}_2^{(\alpha)}(x) = 2(x^2 - \alpha),$$

which, since $\alpha < 0$, has two non-real zeros.

We see already that the case $\alpha < 0$ is very different from the case $\alpha > 0$. This should be expected since, in the one case ($\alpha > 0$), the simple set $\mathcal{H}^\alpha$ consists entirely of polynomials having only simple real zeros while, in the other case ($\alpha < 0$), the simple set $\mathcal{H}^\alpha$ consists entirely of polynomials having only simple zeros, all of which lie on the imaginary axis. The next proposition further demonstrates this difference.

Proposition 163. *Suppose $\alpha < 0$ and $r > 0$. Then the sequence $\left\{r^k\right\}_{k=0}^{\infty}$ is an $\mathcal{H}^\alpha$-CZDS if and only if $r \leq 1$.*

Proof. Suppose $\left\{r^k\right\}_{k=0}^{\infty}$ is an $\mathcal{H}^\alpha$-CZDS and let $T_{\mathcal{H}^\alpha}$ denote the $T_{\mathcal{H}^\alpha}$-operator associated with the sequence $\left\{r^k\right\}_{k=0}^{\infty}$. Then, since

$$p(x) = \mathcal{H}_2^{(\alpha)}(x) + \alpha\mathcal{H}_0^{(\alpha)}(x) = (x^2 - \alpha) + \alpha(1) = x^2$$

has only real zeros, we have

$$T_{\mathcal{H}^\alpha}[p(x)] = r^2\mathcal{H}_2^{(\alpha)}(x) + \alpha\mathcal{H}_0^{(\alpha)}(x) = r^2(x^2 - \alpha) + \alpha(1) = r^2x^2 + \alpha(1 - r^2) \in \mathcal{L} - \mathcal{P},$$

which, since $\alpha < 0$, occurs if and only if $1 - r^2 \geq 0$. Therefore $r \leq 1$.

Conversely, suppose $0 < r \leq 1$. If $r = 1$, then the sequence $\left\{r^k\right\}_{k=0}^{\infty} = \{1\}_{k=0}^{\infty}$ is clearly an $\mathcal{H}^\alpha$-CZDS. Thus, we may assume $0 < r < 1$. Suppose

$$p(x) = \sum_{k=0}^{m} a_k\mathcal{H}_k^{(\alpha)}(x) \in \mathbb{R}[x]$$

and, for any real numbers A and B, define the polynomial $f_{A,B}(x)$ by

$$f_{A,B}(x) = Ap(x) - xp'(x) - Bp''(x)$$

By Proposition 68, the inequality $Z_C\big(f_{A,B}(x)\big) \leq Z_C\big(p(x)\big)$ holds whenever $A, B \geq 0$ and $A \geq m = \deg(q)$. Since we have assumed $\alpha < 0$, the inequality

$$Z_C\big(- Ap(x) + xp'(x) - \alpha p''(x)\big) = Z_C\big(- f_{A,-\alpha}(x)\big) \leq Z_C\big(p(x)\big) \tag{6.8}$$

holds whenever $A \geq m$. From the relation (6.7) we have

$$(-A + xD - \alpha D^2)\left[\mathcal{H}_k^{(\alpha)}(x)\right] = (-A + k)\mathcal{H}_k^{(\alpha)}(x) \qquad (k = 0, 1, 2, \ldots).$$

Thus, we may rewrite inequality (6.8) as

$$Z_C\left(\sum_{k=0}^{m} a_k(-A + k)\mathcal{H}_k^{(\alpha)}(x)\right) \leq Z_C\left(\sum_{k=0}^{m} a_k\mathcal{H}_k^{(\alpha)}(x)\right) \qquad (A \geq m). \tag{6.9}$$

Note that the sequence $\left\{r^k\right\}_{k=0}^{\infty}$ can be interpolated by the function

$$\varphi(x) = r^x = e^{(\ln r)x}$$

which is the uniform limit on compact subsets of $\mathbb{C}$ of the sequence of polynomials (see Lemma 16)

$$g_n\left(\frac{x}{n}\right) = \left(1 + \frac{(\ln r)x}{n}\right)^n \qquad (n = 0, 1, 2, \ldots).$$

Since we have assumed $0 < r < 1$, we have $\ln r < 0$ and, therefore, the zeros of $g_n(x)$ become arbitrarily large and positive as n tends to infinity. By iteration of the inequality (6.9), we see that there exists an integer N such that the inequality

$$Z_C\left(\sum_{k=0}^{m} a_k g_n\left(\frac{k}{n}\right)\mathcal{H}_k^{(\alpha)}(x)\right) \leq Z_C\left(\sum_{k=0}^{m} a_k\mathcal{H}_k^{(\alpha)}(x)\right)$$

holds whenever $n \geq N$. Since the sequence of polynomials

$$\left\{\sum_{k=0}^{m} a_k g_n\left(\frac{k}{n}\right)\mathcal{H}_k^{(\alpha)}(x)\right\}_{n=1}^{\infty}$$

converge uniformly on compact subsets of $\mathbb{C}$ to the polynomial $\sum_{k=0}^{m} a_k r^k \mathcal{H}_k^{(\alpha)}(x)$, we have by Hurwitz' theorem,

$$Z_C\left(\sum_{k=0}^{m} a_k r^k \mathcal{H}_k^{(\alpha)}(x)\right) \leq Z_C\left(\sum_{k=0}^{m} a_k \mathcal{H}_k^{(\alpha)}(x)\right)$$

as desired. □

We summarize the results regarding geometric sequences and generalized Hermite polynomials in the following theorem.

Remark 164. Suppose $r \in \mathbb{R} \setminus \{0\}$. Then we have shown the following.

1. If $\alpha > 0$ then $\left\{r^k\right\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-CZDS if and only if $|r| \geq 1$ (Corollary 129).

2. If $\alpha = 0$ then $\left\{r^k\right\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-CZDS for all $r \neq 0$ (Remark 59).

3. If $\alpha < 0$ then $\left\{r^k\right\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-CZDS if and only if $|r| \leq 1$ (Proposition 163).

The symmetry of these results is quite remarkable and leads one to wonder whether or not every CZDS which is *not* an $\mathcal{H}^{\alpha}$-CZDS for $\alpha > 0$ must be an $\mathcal{H}^{\alpha}$-CZDS for $\alpha < 0$. This leads us to the following problem.

Problem 165. If $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence, then does there exists a non-zero real constant α such that $\{\gamma_k\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-multiplier sequence? Similarly, if $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS, then does there exists a non-zero real constant α such that $\{\gamma_k\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-CZDS?

We remark that, if $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence satisfying $0 \leq \gamma_k \leq \gamma_{k+1}$ for all k, then $\{\gamma_k\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-multiplier sequence for any $\alpha > 0$ (see Theorem 152 and Lemma 161). Furthermore, if $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS which can be interpolated by a polynomial, then $\{\gamma_k\}_{k=0}^{\infty}$ is an $\mathcal{H}^{\alpha}$-CZDS for any $\alpha > 0$, since the polynomials which interpolate CZDS are the same as those that interpolate H-CZDS (Compare Theorems 63 and 111).

To captivate the reader, we state the following special case of the previous problem.

Problem 166. Suppose $\alpha < 0$. Is the CZDS $\left\{\frac{1}{k!}\right\}_{k=0}^{\infty}$ an an $\mathcal{H}^{\alpha}$-CZDS?

We remark that, since the function $1/\Gamma(x+1)$ has zeros at each of the negative integers, we cannot appeal to Proposition 68, as we did in the proof of Proposition 163, and so this problem remains open.

6.3 Laguerre Polynomials

We have seen that there are several interesting results regarding H-multiplier sequences, where H is the set of Hermite polynomials. Recall that the Hermite polynomials form an orthogonal set over the interval $(-\infty, \infty)$ with respect to the weight function $\exp(-x^2)$ (see equation (2.8)). In this section we will investigate Q-multiplier sequences for another orthogonal set of polynomials. The results of this section are by no means complete, but we include them so that we can compare and contrast the some of the multiplier sequences for two different orthogonal polynomial sets.

The set of Laguerre polynomials $L = \{L_n(x)\}_{n=0}^{\infty}$ are explicitly defined by the equation (see [27, p. 213])

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} \qquad (n = 0, 1, 2, \dots). \tag{6.10}$$

For the convenience of the reader, we list the first few Laguerre polynomials here.

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1 - x \\ L_2(x) &= \frac{1}{2}(x^2 - 4x + 2) \\ L_3(x) &= -\frac{1}{3!}(x^3 - 9x^2 + 18x - 6) \\ L_4(x) &= \frac{1}{4!}(x^4 - 16x^3 + 72x^2 - 96x + 24) \\ L_5(x) &= -\frac{1}{5!}(x^5 - 25x^4 + 200x^3 - 600x^2 + 600x - 120) \end{aligned}$$

From equation (6.10) it is easy to see that the set of Laguerre polynomials L forms a simple set of polynomials. Furthermore, it can be shown that the Laguerre polynomials satisfy (see [27, p. 214])

$$\int_0^\infty \exp(-x) L_n(x) L_m(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Thus, the Laguerre polynomials form an orthogonal set over the interval $[0, \infty)$ with respect to the weight function $\exp(-x)$. One can also show that the Laguerre polynomials satisfy Laguerre's differential equation (see [27, p. 214])

$$nL_n(x) = (x-1)L_n'(x) - xL_n''(x) \qquad (n = 0, 1, 2, \dots). \tag{6.11}$$

Let us now demonstrate that there are non-trivial L-multiplier sequences.

Lemma 167. *If the real polynomial $p(x)$ has only real zeros, then the real polynomial $(x-1)p(x) - xp'(x)$ also has only real zeros.*

Proof. By Laguerre's theorem (Theorem 60), the operator $T = 1 + xD$ preserves reality of zeros and, therefore, maps the entire Laguerre-Pólya class into itself (see Proposition 71), i.e., if $\varphi(x) \in \mathcal{L} - \mathcal{P}$, then $\varphi(x) + x\varphi'(x) \in \mathcal{L} - \mathcal{P}$. In particular, since the function $\varphi(x) = -p(x)e^{-x}$ belongs to the Laguerre-Pólya class,

$$\begin{aligned} T[\varphi(x)] &= -p(x)e^{-x} + x(p(x)e^{-x} - p'(x)e^{-x}) \\ &= e^{-x}\big((x-1)p(x) - xp'(x)\big) \in \mathcal{L} - \mathcal{P}. \end{aligned}$$

Therefore, $(x-1)p(x) - xp'(x)$ has only real zeros. □

Proposition 168. *If the polynomial*

$$p(x) = \sum_{k=0}^{n} a_k L_k(x)$$

has only real zeros, then

$$q(x) = \sum_{k=0}^{n} k a_k L_k(x) \in \mathcal{L} - \mathcal{P}.$$

That is to say, the sequence $\{k\}_{k=0}^{\infty}$ *is an L-multiplier sequence.*

Proof. Let T_L denote the T_L operator associated with the sequence $\{k\}_{k=0}^{\infty}$. We claim that $T_L = (x-1)D - xD^2$. Indeed, since the Laguerre polynomials satisfy Laguerre's differential equation (6.11), we have

$$\big((x-1)D - xD^2\big)[L_k(x)] = kL_k(x) = T_L[L_k(x)] \qquad (k = 0, 1, 2, \dots).$$

Since we have assumed $p(x)$ has only real zeros we have, by Rolle's theorem, that $p'(x)$ also has only real zeros (or, perhaps, is identically zero). Therefore, by Lemma

167, we have

$$q(x) = (x-1)p'(x) - xp''(x) \in \mathcal{L}-\mathcal{P}.$$

$\square$

Let us now employ the Hermite-Poulain Theorem (Theorem 37) and Laguerre's theorem (Theorem 60) to demonstrate the existence of another L-multiplier sequence.

Proposition 169. *If the polynomial*

$$p(x) = \sum_{k=0}^{n} a_k L_k(x)$$

has only real zeros, then the polynomial

$$q(x) = \sum_{k=0}^{n} (k+1) a_k L_k(x)$$

also has only real zeros. That is to say, the sequence $\{k+1\}_{k=0}^{\infty}$ is an L-multiplier sequence.

Proof. Let T_L denote the T_L-operator associated with the sequence $\{k+1\}_{k=0}^{\infty}$. We claim that $T_L = 1 + (x-1)D - xD^2$. Indeed, since the Laguerre polynomials satisfy Laguerre's differential equation (6.11), we have

$$\left(1 + (x-1)D - xD^2\right) = (1+k)L_k(x) = T_L[L_k(x)] \qquad (k = 0, 1, 2, \dots).$$

Since we have assumed $p(x)$ has only real zeros we have, by the Hermite-Poulain Theorem (Theorem 37), that the polynomial $p(x) - p'(x)$ has only real zeros. Thus, by Laguerre's theorem (Theorem 60),

$$\begin{aligned}(1+xD)[p(x)-p'(x)] &= \big(p(x)-p'(x)\big)+x\frac{d}{dx}(p(x)-p'(x)) \\ &= p(x)+(x-1)p'(x)-xp''(x) \\ &= q(x)\end{aligned}$$

has only real zeros, as desired. □

So far, we have demonstrated the existence of two L-multiplier sequences, each of which are also H-multiplier sequences. We will now show that there are a large number of H-multiplier sequences which are *not* L-multiplier sequences.

Proposition 170. *If $\alpha > 1$, then the sequence $\{\alpha+k\}_{k=0}^{\infty}$ is not an L-multiplier sequence.*

Proof. Fix $\alpha > 1$ and let T_L denote the T_L-operator associated with the sequence $\{\alpha+k\}_{k=0}^{\infty}$. We claim that $T_L = \alpha + (x-1)D - xD^2$. Indeed, since the Laguerre polynomials satisfy Laguerre's differential equation (6.11), we have

$$\big(\alpha+(x-1)D-xD^2\big)[L_k(x)] = (\alpha+k)L_k(x) = T_L[L_k(x)] \qquad (k=0,1,2,\dots).$$

For any integer $n \geq 2$ we, therefore, have

$$\begin{aligned}T_L\left[(x+n)^n\right] &= \alpha\left[(x+n)^n\right]+(x-1)D\left[(x+n)^n\right]-xD^2\left[(x+n)^n\right] \\ &= \alpha(x+n)^n+(x-1)n(x+n)^{n-1}-xn(n-1)(x+n)^{n-2} \\ &= (x+n)^{n-2}\big((\alpha+n)x^2+2n\alpha x+(\alpha-1)n^2\big).\end{aligned}$$

The discriminant of the quadratic polynomial $(\alpha+n)x^2+2n\alpha x+(\alpha-1)n^2$ is

$$4n^2((1-\alpha)n+\alpha). \tag{6.12}$$

Since we have assumed $\alpha > 1$, we have $1 - \alpha < 0$ and, therefore, we may pick n large enough that the discriminant (6.12) is negative. Thus, for any such n, $T_L[(x+n)^n] \notin \mathcal{L} - \mathcal{P}$ and, therefore the sequence $\{\alpha + k\}_{k=0}^{\infty}$ is not an L-multiplier sequence.

$\square$

Remark 171. Again we emphasize that this situation is vastly different from that of the Hermite polynomials H. Indeed, *every* sequence $\{\alpha + k\}_{k=0}^{\infty}$ with $\alpha \geq 0$ is an H-multiplier sequence.

We are inclined to believe that the following problem can be answered in the affirmative.

Problem 172. Is it true that the sequence $\{\alpha + k\}_{k=0}^{\infty}$ is an L-multiplier sequence if and only if $0 \leq \alpha \leq 1$?

Remark 173. We remark that if $\alpha < 0$, then $\{\alpha + k\}_{k=0}^{\infty}$ is not a multiplier sequence and, therefore, cannot be an L-multiplier sequence. Thus, to answer Problem 172 in the affirmative, it only remains to show that it is true for $0 < \alpha < 1$.

6.4 Q-Multiplier Sequences Which Coincide With Multiplier Sequences

In this section we will characterize all simple sets Q whose multiplier sequences coincide with the classical multiplier sequences. First we will need several preparatory results.

Lemma 174. *Let $n \geq 1$ be an integer and let $q(x)$ be a polynomial of degree $n+1$. If, for every $\alpha, \beta \in \mathbb{R}$, the polynomial*

$$f_{\alpha,\beta}(x) = \alpha x^n + \beta q(x) \tag{6.13}$$

has only real zeros, then x^{n-1} divides $q(x)$.

Proof. If $n = 1$ then $x^{n-1} = 1$ clearly divides $q(x)$. Suppose $n > 1$ and write

$$q(x) = a_{n+1}x^{n+1} + a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

By hypothesis,

$$f_{\alpha,1}(x) = a_{n+1}x^{n+1} + (\alpha + a_n)x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has only real zeros for any $\alpha \in \mathbb{R}$. Thus, for any $\alpha \in \mathbb{R}$, the polynomial

$$h_\alpha(x) = x^{n+1} f_{\alpha,1}\left(\frac{1}{x}\right) = a_{n+1} + (\alpha + a_n)x + a_{n-1}x^2 + \cdots + a_0 x^{n+1}$$

and, therefore, its derivative

$$h'_\alpha(x) = \alpha + a_n + 2a_{n-1}x + \cdots + a_0(n+1)x^n$$

has only real zeros. But this is only possible if $\deg(h'_\alpha(x)) \leq 1$. Thus,

$$a_0 = a_1 = \cdots = a_{n-2} = 0$$

and so x^{n-1} divides $q(x)$. $\square$

In particular, the function $f(x)$ given by equation (6.13) in Lemma 174 has a zero at the origin of multiplicity greater than or equal to $n-1$. The next lemma demonstrates that the two remaining zeros of $f(x)$ are either separated by, or coincident with, the origin.

Lemma 175. *Let $n \geq 1$ be an integer and let $q(x)$ be a polynomial of degree $n+1$. If, for every $\alpha, \beta \in \mathbb{R}$, the polynomial*

$$f_{\alpha,\beta}(x) = \alpha x^n + \beta q(x)$$

has only real zeros, then

$$q(x) = cx^{n-1}(x-a)(x-b), \tag{6.14}$$

where a, b, c are real constants, $c \neq 0$, and $ab \leq 0$.

Proof. That $q(x)$ has the form (6.14) is an immediate consequence of Lemma 174 and the fact that $f_{0,1}(x) = q(x)$ has only real zeros. It only remains to show that $ab \leq 0$. By hypothesis,

$$f_{a+b,1/c}(x) = (a+b)x^n + x^{n-1}(x-a)(x-b) = x^{n-1}\big(x^2 + ab\big)$$

has only real zeros. Thus $ab \leq 0$, as desired. $\square$

If two real entire functions f and g have the property that any linear combination (over $\mathbb{R}$) of f and g has only real zeros, then the pair f and g is called a *generalized real pair* (see [20, p. 315]). The property of being a generalized real pair is quite strong (see [20, Chapter VII]). In the next lemma, it is shown that every pair of consecutive polynomials of a simple set Q form a generalized real pair, provided we assume a certain class of sequences are Q-multiplier sequences.

Lemma 176. *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of real monic polynomials. If, for all integers $m \geq 1$, the sequence*

$$\{k(k-1)(k-2)\cdots(k-m+1)\}_{k=0}^{\infty}$$

is a Q-multiplier sequence, then for any integer $n \geq 0$ and any $\alpha, \beta \in \mathbb{R}$,

$$f_{\alpha,\beta}(x) = \alpha q_n(x) + \beta q_{n+1}(x) \in \mathcal{L} - \mathcal{P}. \tag{6.15}$$

Proof. It will first be shown that each polynomial in the simple set Q has only real zeros, and this will establish the result when α or β (or both) are equal to zero. Since $q_0(x)$ is a (non-zero) constant function and $q_1(x)$ is a linear function, each of these polynomials have only real zeros. Fix an integer $n \geq 2$ and let

$$p(x) = \sum_{k=0}^{n} a_k q_k(x) \qquad (a_n \neq 0)$$

be a polynomial which has only real zeros. Applying the Q-multiplier sequence

$$\{k(k-1)(k-2)\cdots(k-n+1)\}_{k=0}^{\infty}$$

to $p(x)$, we see that $a_n n! q_n(x)$, and therefore $q_n(x)$, has only real zeros. Thus, for any integer $n \geq 0$, $q_n(x)$ has only real zeros.

If α and β are non-zero, then it suffices to show that $f_{\gamma,1}(x)$ has only real zeros, where $\gamma = \dfrac{\alpha}{\beta}$. Write

$$q_{n+1}(x) = x^{n+1} + b_n x^n + b_{n-1}x^{n-1} + \cdots b_0$$

and

$$q_n(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots c_0.$$

With this notation, we have that the polynomial

$$\begin{aligned}
h(x) &= \left(x + \frac{b_n + \gamma(n+1)}{n+1}\right)^{n+1} \\
&= x^{n+1} + \left(b_n + \gamma(n+1)\right) x^n + \sum_{k=0}^{n-1} d_k x^k \\
&= q_{n+1}(x) + \gamma(n+1) q_n(x) + \sum_{k=0}^{n-1} e_k q_k(x),
\end{aligned}$$

where the d_k's and e_k's are appropriate real constants, has only real zeros. Applying the Q-multiplier sequence $\{k(k-1)\cdots(k-n+1)\}_{k=0}^{\infty}$ to $h(x)$, we see that

$$\begin{aligned}
(n+1)!q_{n+1}(x) + \gamma(n+1)n!q_n(x) &= (n+1)!\left[\gamma q_n(x) + q_{n+1}(x)\right] \\
&= (n+1)!f_{\gamma,1}(x)
\end{aligned}$$

has only real zeros. Therefore, $f_{\gamma,1}(x)$ has only real zeros, as desired $\square$

In the next proposition, we only assume that two types of sequences are multiplier sequences for a given simple set Q which consists entirely of monic polynomials. Amazingly, this is enough to conclude that the simple set Q is actually the standard basis.

Proposition 177. *Let $Q = \{q_k(x)\}_{k=0}^{\infty}$ be a simple set of monic polynomials satisfying $q_0(x) = 1$ and $q_1(x) = x$. If, for all integers $m \geq 1$, the sequence*

$$\{k(k-1)(k-2)\cdots(k-m+1)\}_{k=0}^{\infty} \tag{6.16}$$

is a Q-multiplier sequence, and if there exists an open interval I containing the origin such that, for all $r \in I$, the sequence

$$\left\{r^k\right\}_{k=0}^{\infty} \tag{6.17}$$

is a Q-multiplier sequence, then Q is the standard basis $\{x^k\}_{k=0}^{\infty}$.

Proof. It will first be shown that $q_2(x) = x^2$. By Lemma 176, any linear combination

$$\alpha q_1(x) + \beta q_2(x) = \alpha x + \beta q_2(x) \qquad (\alpha, \beta \in \mathbb{R})$$

has only real zeros. Thus, by Lemma 175, we can write the monic quadratic polynomial $q_2(x)$ in the form

$$q_2(x) = x^2 + bx + c \qquad (b, c \in \mathbb{R}; \quad c \leq 0).$$

The polynomial

$$h(x) = \left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4} = q_2(x) + \left(\frac{b^2 - 4c}{4}\right) q_0(x)$$

has only real zeros. For any non-zero $r \in I$, the sequence $\{r^{k-2}\}_{k=0}^{\infty}$ is a classical multiplier sequence. Thus, for any non-zero $r \in I$, the polynomial

$$\widehat{h}_r(x) = q_2(x) + \left(\frac{b^2 - 4c}{4}\right) \frac{1}{r^2} q_0(x) = x^2 + bx + c + \left(\frac{b^2 - 4c}{4r^2}\right)$$

has only real zeros. Since we have assumed $c \leq 0$, we have $b^2 - 4c \geq 0$. If $b^2 - 4c > 0$, then one could choose r sufficiently small so that $\widehat{h}_r(x)$ has non-real zeros, a contradiction. Therefore, $b^2 - 4c = 0$ and, since b^2 and $-4c$ are both non-negative, $b = 0 = c$. That is to say, $q_2(x) = x^2$ as desired.

Now let $n \geq 2$ be an integer and suppose $q_k(x) = x^k$ for $k = 0, 1, 2, \ldots, n$. It will be shown that $q_{n+1}(x) = x^{n+1}$. By Lemma 176, every linear combination,

$$\alpha q_n(x) + \beta q_{n+1}(x) = \alpha x^n + \beta q_{n+1}(x) \qquad (\alpha, \beta \in \mathbb{R})$$

has only real zeros. Therefore, by Lemmas 174 and 175, we can write the monic $(n+1)^{st}$ degree polynomial $q(x)$ as

$$q_{n+1}(x) = x^{n+1} + bx^n + cx^{n-1} \qquad (b, c \in \mathbb{R}; \quad c \leq 0).$$

The polynomial

$$f(x) = \left(x + \frac{b}{n+1}\right)^{(n+1)},$$

which has only real zeros, can be written in the form

$$f(x) = q_{n+1}(x) + \left(\frac{n(n+1)}{2} \cdot \frac{b^2}{(n+1)^2} - c\right) q_{n-1}(x) + \sum_{k=0}^{n-2} a_k q_k(x).$$

For any non-zero $r \in I$, the sequence

$$\left\{r^{k-(n+1)} k(k-1)(k-2)\cdots(k-(n-2))\right\}_{k=0}^{\infty}$$

is a Q-multiplier sequence. Therefore, for any non-zero $r \in I$, the polynomial

$$\begin{aligned}
\widehat{f}_r(x) &= \frac{(n+1)!}{2!} q_{n+1}(x) + \left(\frac{n(n+1)}{2} \cdot \frac{b^2}{(n+1)^2} - c\right) \frac{(n-1)!}{r^2} q_{n-1}(x) \\
&= \frac{(n+1)!}{2!} \left[x^2 + bx + c + \frac{1}{r^2}\left(\frac{b^2}{(n+1)^2} - \frac{2c}{(n+1)!}\right)\right] x^{n-1} \\
&= \frac{(n+1)!}{2!} \left[x^2 + bx + c + \frac{1}{r^2(n+1)^2}\left(b^2 - \frac{n+1}{n!} 2c\right)\right] x^{n-1}
\end{aligned}$$

has only real zeros. Since $c \leq 0$, it follows that $b^2 - \frac{n+1}{n!}2c \geq 0$. If $b^2 - \frac{n+1}{n!}2c > 0$, then one could choose r sufficiently small so that $\widehat{f}_r(x)$ has non-real zeros, a contradiction. Thus $b^2 - \frac{n+1}{n!}2c = 0$ and, since both b^2 and $-\frac{n+1}{n!}2c$ are non-negative, $b = 0 = c$. That is to say, $q_{n+1}(x) = x^{n+1}$ as desired. □

Remark 178. It should be noted that both types of sequences in the hypotheses of Proposition 177 are essential to the conclusion. To see this, we note that the generalized Hermite polynomials $\mathcal{H}^{(\alpha)}$ form a simple set of monic polynomials which satisfy $\mathcal{H}_0^{(\alpha)}(x) = 1$ and $\mathcal{H}_1^{(\alpha)}(x) = x$.

If $\alpha > 0$, then any sequence of the form (6.16) is an $\mathcal{H}^{(\alpha)}$-multiplier sequences (see Lemma 161 and Proposition 116), and $\mathcal{H}^{(\alpha)}$ is clearly not equal to the standard basis when $\alpha > 0$. Furthermore, we note that Proposition 177 is not contradicted, since sequences of the form (6.17) are not $\mathcal{H}^{(\alpha)}$-multiplier sequences for any $r \in (-1, 1)$ (see Remark 164).

Similarly, if $\alpha < 0$, then any sequence of the form (6.17) is an $\mathcal{H}^{(\alpha)}$-multiplier sequence for any $r \in [-1, 1]$ (see Remark 164), and again it is clear that $\mathcal{H}^{(\alpha)}$ is not equal to the standard basis when $\alpha < 0$. Again we note that Proposition 177 is not contradicted, since the sequence $\{k\}_{k=0}^{\infty}$ is not an $\mathcal{H}^{(\alpha)}$-multiplier sequence (see Example 162).

We are now in a position to completely characterize all simple sets Q whose multiplier sequences coincide with the classical multiplier sequences.

Theorem 179. *Suppose $Q = \{q_k(x)\}_{k=0}^{\infty}$ is a simple set of polynomials which has the property that every multiplier sequence is also a Q-multiplier sequence. Then there exists a sequence of non-zero real numbers $\{c_k\}_{k=0}^{\infty}$ and a constant $\beta \in \mathbb{R}$ such that $q_k(x) = c_k\,(x + \beta)^k$ for all integers $k \geq 0$.*

Proof. Suppose the polynomials of the simple set Q can be written as

$$q_k(x) = \sum_{j=0}^{k} a_{k,j} x^j \qquad (k = 0, 1, 2, \dots).$$

Note that, since Q is a simple set, $a_{k,k} \neq 0$ for all integers $k \geq 0$. Define the simple set of polynomials $\widehat{Q} = \{\widehat{q}_k(x)\}$ by

$$\widehat{q}_k(x) = \frac{1}{a_{k,k}} q_k\left(x - \frac{a_{1,0}}{a_{1,1}}\right) \qquad (k = 0, 1, 2, \ldots). \tag{6.18}$$

By hypothesis, the multiplier sequences

$$\{k(k-1)(k-2)\cdots(k-m+1)\}_{k=0}^{\infty} \qquad (m = 1, 2, 3, \ldots)$$

and

$$\left\{r^k\right\}_{k=0}^{\infty} \qquad (r \in \mathbb{R} \setminus \{0\})$$

are Q-multiplier sequences and, by Lemma 156, are therefore $\widehat{Q}$-multiplier sequences as well. Furthermore, each polynomial $\widehat{q}_k(x)$ is monic and we also have

$$\widehat{q}_0(x) = \frac{1}{a_{0,0}} q_0\left(x - \frac{a_{1,0}}{a_{1,1}}\right) = \frac{1}{a_{0,0}} a_{0,0} = 1$$

and

$$\widehat{q}_1(x) = \frac{1}{a_{1,1}}\left(a_{1,1}\left(x - \frac{a_{1,0}}{a_{1,1}}\right) + a_{1,0}\right) = x.$$

Therefore, by Proposition 177, the simple set $\widehat{Q}$ is equal to the standard basis. That is to say, $\widehat{q}_k(x) = x^k$ for all integers $k \geq 0$. In light of the definition of the polynomials $\widehat{q}_k(x)$ given in equation (6.18), we have

$$\frac{1}{a_{k,k}} q_k\left(x - \frac{a_{1,0}}{a_{1,1}}\right) = x^k \qquad (k = 0, 1, 2, \ldots).$$

Therefore, we have

$$q_k(x) = a_{k,k}\left(x + \frac{a_{1,0}}{a_{1,1}}\right)^k \qquad (k = 0, 1, 2, \ldots),$$

as desired. $\square$

Remark 180. By Lemma 156, a sequence is a multiplier sequence if and only if it is a Q-multiplier sequence, where the simple set $Q = \{q_k(x)\}_{k=0}^{\infty}$ is defined by

$$q_k(x) = c_k(x+\beta)^k \qquad (c_k \in \mathbb{R} \setminus \{0\};\ \beta \in \mathbb{R}).$$

Theorem 179 shows that these are the *only* simple sets whose multiplier sequences coincide with the classical multiplier sequences.

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